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THE SINGLE SHOOTING METHOD APPLIED TO AN ORBIT COPLANAR TRANSFER

Konstantin Metodiev

Space Research and Technology Institute – Bulgarian Academy of Sciences
e-mail: komet@space.bas.bg

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Abstract

In this paper, the Single-shooting method is applied to design a coplanar orbital transfer trajectory for an artificial Earth satellite. The satellite moves in a central force field, taking into account the central body (the Earth) oblateness (J2 zonal coefficient). During transfer trajectory computation, the state vector is propagated numerically, and a state transition matrix is used to determine how small disturbances in the initial velocity vector affect the final position vector. For this purpose, the so-called Hohmann coplanar transfer (altitude change) is used as a validation case. The numerical results are presented graphically and discussed thoroughly.

Introduction

The single shooting method is a numerical approach to solving boundary value problems (BVPs) for ordinary differential equations (ODEs). The method essentially reduces the boundary value problem to a system of ODEs with initial conditions (initial value problem – IVP) that is easier to solve.

The single shooting method is relatively easy to implement. The method works well with problems that are less sensitive to initial conditions, yet difficulties are expected in solving stiff problems, such as divergence or instability during iterations.

Coplanar manoeuvres such as Hohmann, bi-polar, and one-tangent burns have already been thoroughly investigated and formulae developed in many textbooks in Astrodynamics, such as Vallado [1], to name but a few. In particular, Hohmann orbit transfer analysis concerning national security issues has been carried out by Marinov in paper [2].

The current study presents a numerical implementation of the single-shooting method to perform a co-planar Hohmann orbital transfer. Rather than conducting theoretical research, the current study focuses on the numerical algorithm stages to bring the problem to successful completion.

Governing equations

A solution to the following IVP describes a satellite motion in a central gravitational field (two-body problem) expressed in Earth-centred inertial (ECI) reference frame:

$$(1)\qquad \dot{\mathbf{x}} = \begin{Vmatrix}\dot{\mathbf{r}}\\ \dot{\mathbf{v}}\end{Vmatrix} = \begin{Vmatrix}\mathbf{0}_{3x3} & \mathbf{I}_{3x3}\\ -\left(\nabla U + \nabla U_{J2}\right) & \mathbf{0}_{3x3}\end{Vmatrix}\begin{Vmatrix}\mathbf{r}\\ \mathbf{v}\end{Vmatrix} \qquad \begin{matrix} r = \left\|\mathbf{r}\right\|_2 = \sqrt{x^2+y^2+z^2}\\ \mathbf{r}\left(t_0\right)=\mathbf{r}_0 \quad \mathbf{v}\left(t_0\right)=\mathbf{v}_0\end{matrix}$$

where two potential functions U and U_{J2}, taking into account the second zonal harmonic coefficient J2 = 1.08262668E–3, are defined as follows. The Earth is regarded as a potential well, hence the negative sign.

$$(2)\qquad U = -\frac{\mu}{r} \quad U_{J2} = -\frac{\mu}{r}J_2\left(\frac{R_\oplus}{r}\right)^2\left[\frac{3}{2}\left(\frac{z}{r}\right)^2-\frac{1}{2}\right]$$

The inclusion of J2 accounts for the Earth's deviation from perfect sphericity due to the equatorial bulge. The J2 perturbation, in turn, affects the right ascension of the ascending node Ω (line of nodes precession) and the argument of perigee ω.

The gradient of the potential function ∇UJ2 was computed symbolically by the SymPy package implemented in GNU Octave. For example, the partial derivative $\partial U_{J2}/\partial x$ was reported as shown in Fig. 1.

```
Command Window
>> pkg load symbolic
>> syms x y z r U J2 RE mu
>> r = sqrt(x^2 + y^2 + z^2);
>> U = -mu/r*J2*(RE/r)^2*(3/2*(z/r)^2 - 1/2);
>> dUdx = diff(U,x)
dUdx = (sym)

                                     /         2          \
                              2      |      3*z          1|
                       3*J2*RE *mu*x*|----------------- - -|
          2        2                 |  / 2     2    2\   2|
   3*J2*RE *mu*x*z                   \2*\x  + y  + z /    /
  ----------------- + ------------------------------------
                 7/2                            5/2
  / 2    2    2\                  / 2    2    2\
  \x  + y  + z /                  \x  + y  + z /

>> |
Command Window   Documentation   Variable Editor   Editor
Profiler
```

Fig. 1. SymPy exemplary session

After computing gradients ∇U and ∇U_{J2} and replacing them in (1), the following formulae for acceleration vector d**v**/dt components are obtained:

$$a_x = -\frac{\mu x}{r^3}\left[1+\frac{3}{2}J_2\left(\frac{R_\oplus}{r}\right)^2\left[1-5\left(\frac{z}{r}\right)^2\right]\right]$$

$$a_y = \frac{y}{x}a_x \tag{3}$$

$$a_z = -\frac{\mu z}{r^3}\left[1+\frac{3}{2}J_2\left(\frac{R_\oplus}{r}\right)^2\left[3-5\left(\frac{z}{r}\right)^2\right]\right]$$

This constitutes a system of 6 first-order ODEs in terms of the state vector **x**.

The state transition matrix $\mathbf{\Phi}(t,t_0)$ is a fundamental term in a linearised celestial mechanics problem. It is used to propagate small deviations in the initial state (a.k.a. differential correction). In the case of non-linear equations

$$\dot{\mathbf{x}}(t) = f(\mathbf{x},t) \tag{4}$$

the perturbed state $\delta\mathbf{x}(t)$ might be derived from initial deviation $\delta\mathbf{x}(t_0)$ as follows

$$\delta\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)\,\delta\mathbf{x}(t_0) \tag{5}$$

Equation (5) is required to compute small deviations $\delta\mathbf{v}_0$ necessary to diminish deviation from the desired position $\delta\mathbf{r}_1$

$$\begin{Vmatrix}\delta\mathbf{r}_1\\ \delta\mathbf{v}_1\end{Vmatrix} = \begin{Vmatrix}\Phi_{rr}(t,t_0) & \Phi_{rv}(t,t_0)\\ \Phi_{vr}(t,t_0) & \Phi_{vv}(t,t_0)\end{Vmatrix}\begin{Vmatrix}\delta\mathbf{r}_0\\ \delta\mathbf{v}_0\end{Vmatrix} \tag{6}$$

The immobility constraint implies $\delta\mathbf{r}_0 = 0$. If the final velocity deviation $\delta\mathbf{v}_1$ is left unconstrained, then the initial velocity is amended in accordance with the formula

$$\delta\mathbf{v}_0 = \Phi_{rv}(t,t_0)^{-1}\,\delta\mathbf{r}_1 \tag{7}$$

Besides the state vector **x**, the state transition matrix $\mathbf{\Phi}$ is propagated too. The initial conditions $\mathbf{x}(t_0)$ can be adapted to obtain a numerical solution x1(t1) near the desired values $\mathbf{x}_d(t_1)$. Hence, a numerical solution to the following IVP of 36 first-order ODEs is sought in addition.

$$\dot{\mathbf{\Phi}}(t,t_0) = \mathbf{A}(t)\mathbf{\Phi}(t,t_0) \quad \mathbf{\Phi}(t_0,t_0) = \mathbf{I} \tag{8}$$

where the block matrix *A* is

$$\text{(9)} \qquad \mathbf{A}(t) = \left\| \begin{matrix} \mathbf{0}_{3x3} & \mathbf{I}_{3x3} \\ \dfrac{\partial \mathbf{a}(\mathbf{x},t)}{\partial \mathbf{r}} & \mathbf{0}_{3x3} \end{matrix} \right\| \qquad \frac{\partial \mathbf{a}(\mathbf{x},t)}{\partial \mathbf{r}} = \left\| \begin{matrix} \dfrac{\partial a_x}{\partial x} & \dfrac{\partial a_x}{\partial y} & \dfrac{\partial a_x}{\partial z} \\ \dfrac{\partial a_y}{\partial x} & \dfrac{\partial a_y}{\partial y} & \dfrac{\partial a_y}{\partial z} \\ \dfrac{\partial a_z}{\partial x} & \dfrac{\partial a_z}{\partial y} & \dfrac{\partial a_z}{\partial z} \end{matrix} \right\|$$

Partial derivatives $\partial\mathbf{a}(\mathbf{x},t)/\partial\mathbf{r}$ are obtained as excessively large by SymPy even after applying simplification. This is the reason why the following formulae derived in thesis [3] have been adopted instead:

$$\text{(10)} \qquad \begin{aligned}
&\frac{\partial a_x}{\partial x} = \frac{\mu}{r^5}\left[3x^2 - r^2 - \frac{3}{2}J_2R_\oplus^5 + \frac{15}{2}\frac{J_2R_\oplus^2}{r^2}\left(x^2+z^2\right) - \frac{105}{2}\frac{J_2R_\oplus^2}{r^4}x^2z^2\right] \\
&\frac{\partial a_x}{\partial y} = \frac{3\mu xy}{r^5}\left(1 + \frac{5}{2}\frac{J_2R_\oplus^2}{r^2} - \frac{35}{2}\frac{J_2R_\oplus^2}{r^4}z^2\right) \\
&\frac{\partial a_x}{\partial z} = \frac{3\mu xz}{r^5}\left(1 + \frac{15}{2}\frac{J_2R_\oplus^2}{r^2} - \frac{35}{2}\frac{J_2R_\oplus^2}{r^4}z^2\right) \\
&\frac{\partial a_y}{\partial x} = \frac{\partial a_x}{\partial y}; \quad \frac{\partial a_y}{\partial y} = \frac{y}{x}\frac{\partial a_x}{\partial y} + \frac{a_x}{x}; \quad \frac{\partial a_y}{\partial z} = \frac{y}{x}\frac{\partial a_x}{\partial z} \\
&\frac{\partial a_z}{\partial x} = \frac{\partial a_x}{\partial z}; \quad \frac{\partial a_z}{\partial y} = \frac{\partial a_y}{\partial z} \\
&\frac{\partial a_z}{\partial z} = \frac{\mu}{r^5}\left[-r^2 + 3\left(z^2 - \frac{3}{2}J_2R_\oplus^5 + \frac{15}{2}\frac{J_2R_\oplus^2}{r^2}z^2 - \frac{35}{2}\frac{J_2R_\oplus^2}{r^4}z^4\right)\right]
\end{aligned}$$

The single-shooting method is depicted schematically in Fig. 2. Given certain initial conditions x(t0), the propagated vector x(t1) differs from the desired value $\mathbf{x}_d(t_1)$ by some error $\delta\mathbf{x}(t_1)$. The error is then backpropagated (6) to work out the initial vector value necessary for achieving the desired outcome. In most cases, it only takes a few iterations to succeed. The process is repeated until the Euclidean norm of the absolute error vector $\|\delta\mathbf{x}(t_1)\|_2$ becomes sufficiently small. Alternative definitions of residual are also acceptable.

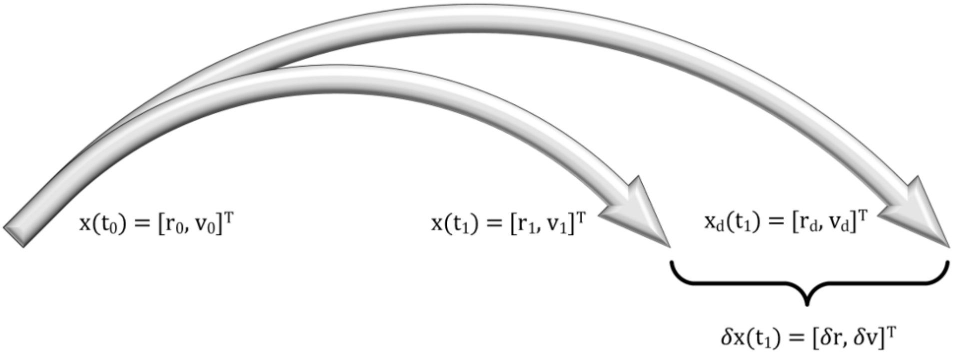

Fig. 2. The single shooting formalism

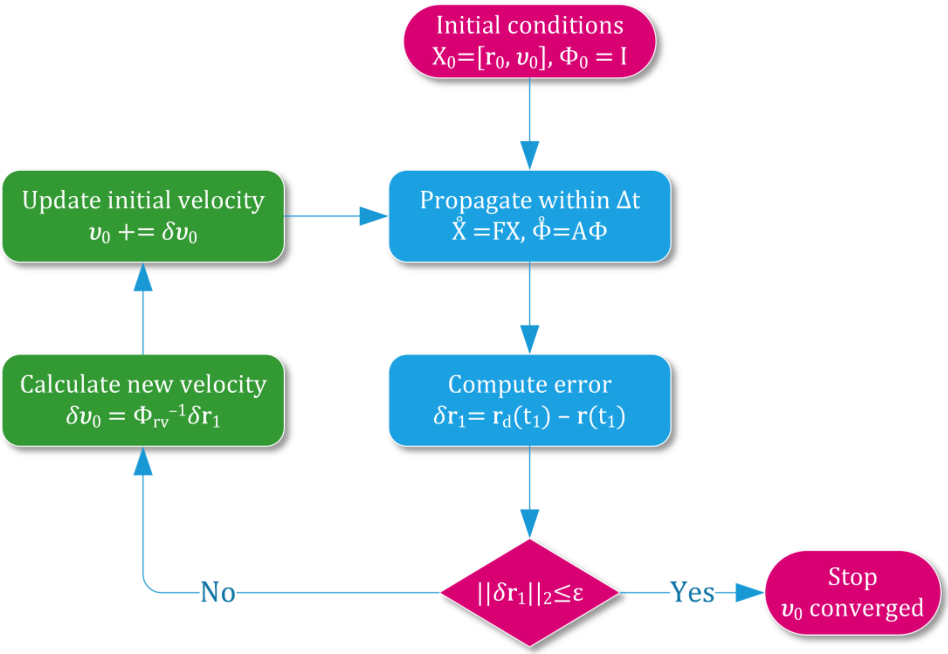

Fig. 3. Computational steps taken by the single shooting method, [4]

The single shooting algorithm is depicted in Fig. 3 as a flowchart adopted from Ref. [4]. To start with, initial values (ECI reference frame) are assigned to the state vector whilst the state transition matrix is initialised with identity one. Both the state vector and the state transition matrix are propagated, and the final position is compared to the desired one. In case of a large difference, the initial velocity is updated according to formula (7). The transfer trajectory is propagated until the error vector $\delta\mathbf{r}_1$ becomes sufficiently small.

The desired position is computed by altering the flight altitude, which is essentially a manoeuvre confined within the orbital plane. It suggests the position vector $\mathbf{r}(t_1)$ is simply enlarged, for example, by a factor $2R_\oplus$

$$\mathbf{r}_d\left(t_1\right) = \frac{\mathbf{r}\left(t_1\right)}{\left\|\mathbf{r}\left(t_1\right)\right\|_2} 2R_\oplus \tag{11}$$

Two transfer trajectories with two bursts are computed: the first burst deviates from the initial orbit, and the second burst fine-tunes what is expected to be a final orbit.

The GNU Octave function ode45 is used in the current project to propagate both the state vector and the state transition matrix. The function implements the explicit Dormand-Prince method of order 4, [5]. It is only capable of solving (systems of non-stiff) first-order ODE(s), which is yet another reason why the governing equations (1) are written in vector form.

Results

Consider the following osculating Keplerian elements as initial conditions:

- Semi-major axis $a = 8639$ km
- Eccentricity $e = 0.1869$
- Inclination $i = 34.28$ deg
- Argument of perigee $\omega = 331.9$ deg
- Right ascension of the ascending node $\Omega = 348.7$ deg
- Mean anomaly $M = 19.22$ deg

Before commencing iterations, these elements must be converted to position and velocity vectors in the ECI reference frame. An excellent tutorial on conversions between $[\mathbf{r}, \mathbf{v}]^T$ ECI vector and Keplerian elements might be found in link [6]. The resulting initial ECI vector $\mathbf{x}_0 = [\mathbf{r}_0, \mathbf{v}_0]^T$ at epoch 0 is (units are km and km/s) $\mathbf{x}_0 = [7021, -1393, 5.459, 1.890, 6.409, 4.537]^T$

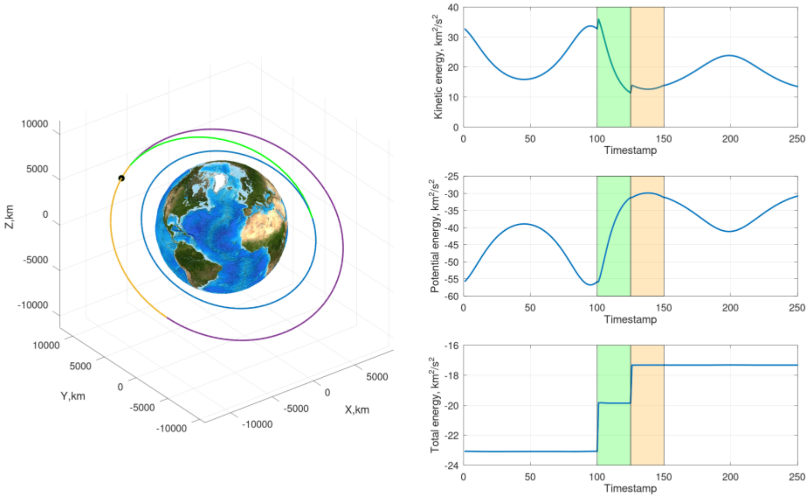

Fig. 4. Numerical results

In Fig. 4, the initial orbit, two transfer trajectories, and final orbit are depicted next to kinetic, potential, and total energy graphs. The total energy remains constant along each orbital segment as expected. Shaded regions (green and orange) make it easier for the reader to discern two consecutive transfer trajectories in the graphs. The potential energy is always negative, for the reference altitude has been chosen at infinity.

In the current case study, the norm 2 of the absolute error approaches zero rapidly. It takes the algorithm only five iterations to converge, as shown in Fig 5.

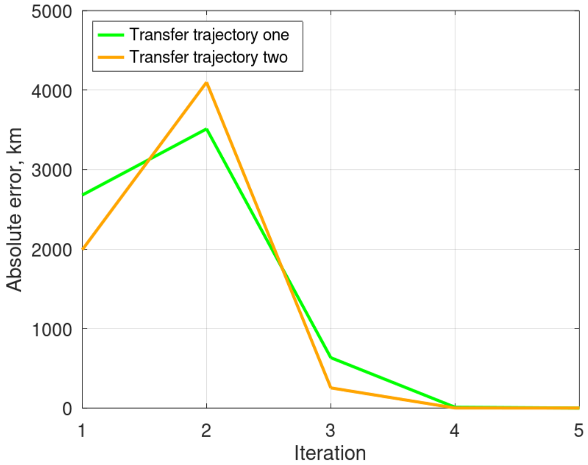

Fig. 5. Numerical solution convergence

Conclusion

The presented implementation of the Single Shooting Method is a prerequisite for performing more complex manoeuvres, including noncoplanar ones. In addition, the developed source code in GNU Octave might be used as a testbed for further project improvement. The **code** might be **downloaded** from link [7].

Multiple shooting and Finite differences are alternatives to the Single shooting method. These methods distribute nonlinearities and improve numerical stability.

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ОРБИТАЛЕН РАВНИНЕН ТРАНСФЕР КАТО ГРАНИЧНА ЗАДАЧА

К. Методиев

Резюме

В настоящата статия е демонстриран „Методът на стрелбата“ за проектиране на преходни равнинни траектории на движение на изкуствен спътник на Земята. Спътникът се движи в централно силово поле с отчитане на сплеснатостта на централното тяло (зонален коефициент J2). По време на пресмятане на преходната траектория се изчисляват както вектора, така и преходната матрица на състоянието, за да се разбере как малки смущения в началната скорост отклоняват крайната точка на траекторията. За тази цел т. нар. „Трансфер на Хохман“ (промяна на височината) е използван за валидация. Числените резултати са представени графично и обсъдени изчерпателно.