

## INVESTIGATION OF THE MOTION OF A SATELLITE, ACCORDING TO GENERAL THEORY OF RELATIVITY

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### **Abstract**

*In this paper, we investigate the energy integral, which is obtained in the problem of the motion of a material point in a central field within the frame of the general theory of relativity. Applied is the method of the small parameter in a combination with the balance method. Derived is a compact formula, describing the trajectory of the motion. This formula gives a correct quantitative description of the basic relativistic effects. We prove the shortening of the major axis of the orbit compared with the case where we do not consider relativistic effects. This result can be useful for analysing the structure of planet systems around massive stars.*

### **Introduction**

The motion of a satellite in a central gravitational field is one of the milestone problems, whose solution imposes the use of the general theory of relativity. Its solution exactly defines the precession angle of Mercury, a problem, which had engaged the theorists for a long time until the arrival of Einstein's theory [1]. After this success, to a great extent the interest for this problem decreases as it is assumed that the major goals for investigating this problem have already been achieved. The fact that the problem is defined using a nonlinear differential equation gives an opportunity for work in this direction. On one hand, in the literature the problem is solved using sensible physical assumptions for the weak influence of a term in the differential equation on the solution, as the results of such an interpretation are being justified [1–2]. In this work we mathematically motivate the application of the small parameter method [3]. We use a method in which the analytical technique for describing the perturbed behaviour is applied to the energy integral. This was we directly define the link between the orbital parameters of the motion and we derive a compact formula for the solution. A result in such representation is that the major axis of the orbit is being shortened in comparison with the case when we do not take into account

relativistic effects. In this work, we follow a mathematical framework set in the problem of Zelmanov and Agakov [2], but we use a different asymptotic method for finding the solution, which aims mathematically rigorous conclusions.

### Mathematical Framework of the Problem

Consider the motion of a planet  $T$  in the radial field of a star  $S$ . We assume that the central object has a spherical shape and the planet can be approximated as a material point. Let note the distance between the two fields with  $R$ , and the polar angle with  $\varphi$ : Fig. 1.

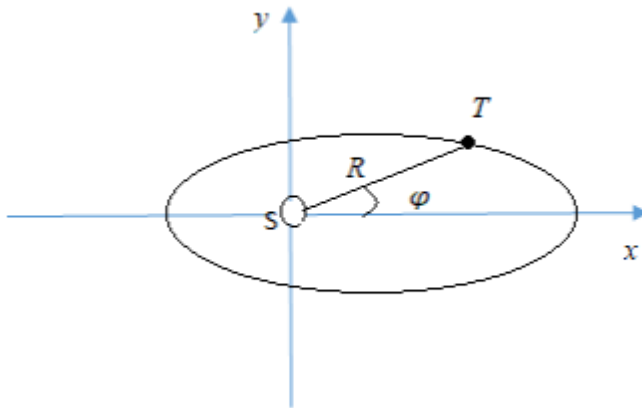


Fig. 1. The motion of a material point in a radial gravitational field

Let assume the mass of the material point to be unity, as well as being negligible compared to the mass  $\mu$  of the body in whose field it is moving. Then, the differential equation of motion of the material point has the following form [2]:

$$(1) \quad \frac{d^2 u}{d\varphi^2} + u = \frac{\gamma\mu}{h^2} + 3\frac{\gamma\mu}{c^2} u^2.$$

In this formula:  $u = \frac{1}{R}$ ,  $c$  is the speed of light in vacuum,  $\gamma$  is the universal gravitational constant,  $h$  is a constant defining the angular momentum, and  $\varphi$  is the angular parameter. We substitute

$$(2) \quad \frac{1}{P} = \frac{\gamma\mu}{h^2}.$$

Changing the variables:

$$(3) \quad \xi = uP.$$

We finally get

$$(4) \quad \frac{d^2\xi}{d\varphi^2} + \xi = 1 + 3\varepsilon\xi^2..$$

The variable

$$(5) \quad \varepsilon = \frac{\gamma\mu}{PC^2}$$

can be interpreted as a small parameter in our further analytical investigations.

Integrating Equation (4) we get

$$(5) \quad \left(\frac{d\xi}{d\varphi}\right)^2 + \xi^2 - 2\xi - 2\varepsilon\xi^3 = 2\varepsilon, \quad \varepsilon = \frac{EP}{\gamma\mu},$$

E is the energy of the system.

### **Finding and asymptotic solution**

We are looking for a solution in a series [4]:

$$(6) \quad \xi = \xi_0 C_0 + \varepsilon \xi_1 + \varepsilon C_1,$$

where  $\xi_0$  and  $\xi_1$  are functions of  $\varphi$ ,  $C_0$  and  $C_1$  are constants. We make the guess that for the angular variable we have:

$$(7) \quad \varphi = \varphi_0 + \varepsilon \varphi_1.$$

In further calculations we will use the formula:

$$(8) \quad \frac{d}{d\varphi} = \frac{d}{d\varphi_0} - \varepsilon \frac{d\varphi_1}{d\varphi_0} \frac{d}{d\varphi_0}.$$

In addition, we write the integral constant as

$$(9) \quad \varepsilon = \varepsilon_0 + \varepsilon \varepsilon_1.$$

For Equation (5) to zeroth order we get

$$(10) \quad \left(\frac{d\xi_0}{d\varphi_0}\right)^2 + \xi_0^2 + C_0^2 + 2\xi_0 C_0 - 2\xi_0 - 2C_0 = 2\epsilon_0.$$

We are looking for a solution in the form

$$(11) \quad \xi_0 = e \cos \varphi_0.$$

It is easy to see that  $C_0 = 1$ . Then for the constant  $e$  we get

$$(12) \quad e^2 = 2\epsilon_0 + 1.$$

The parameter  $e$  is the *eccentricity* of the orbit, and  $P$  – the *focal parameter*. For the orbit to be an ellipse, we should have the following condition:  $-1 < 2\epsilon_0 < 0$ .

For the first approximation of the equation we get:

$$(13) \quad \xi_1 = eB \cos \varphi_0.$$

We look for  $\xi_1$  in the form:

$$(14) \quad \xi_1 = eB \cos \varphi_0.$$

After some calculations we equate the sum of all constants to be zero. The same procedure is carried out for the sum of the coefficients in front of the periodic functions  $\cos \varphi_0$  and  $\cos 2 \varphi_0$ . The calculations are carried out with precision up to  $\cos 2 \varphi_0$ . Then for the parameters we get:

$$\begin{aligned} \epsilon_1 &= -1, \\ \frac{d\varphi_1}{d\varphi_0} &= 3, \\ B &= 3, \\ C_1 &= 3 \left(1 + \frac{e^2}{4}\right). \end{aligned}$$

Then, we substitute:

$$(15) \quad n = 1 - \frac{3\gamma\mu}{c^2 P}.$$

The final equation for  $\xi$  is:

$$(16) \quad \xi = \frac{1}{n} + \frac{e}{n} \cos n\varphi + O\left(\frac{3\gamma\mu}{c^2 P} e^2\right).$$

We can readily write the equation for the trajectory of the satellite:

$$(17) \quad R = \frac{\tilde{P}}{1+e \cos n\varphi}, \tilde{P} = Pn.$$

From this formula it is clear, that the motion on the orbit includes precession but we can also note the shortening of the major axes of the orbits of space objects, which does not depend on the distance between the planet and the central object, but only on the gravity radius of the central object: Table 1.

*Table 1. Relativistic effects*

Precession: $\Delta\omega$	Shortening of the major axis of the elliptical trajectory: $\Delta P$
$\frac{6\pi\gamma\mu}{c^2 P}$	$\frac{3\gamma\mu}{c^2}$
Shortening of the major axes of the orbits for all planets in the Solar system	4.44 $10^3$ m

The value we find for the precession coincides with the value derived analytically using other methods.

We can easily find the equation for energy, including the orbital elements [5]:

$$(18) \quad E = -\frac{\gamma\mu}{2a} - \frac{h^2}{(1-e^2)^3 c^2} \frac{\gamma\mu}{a^3}, P = a(1 - e^2).$$

## Conclusion

The nonlinear character of the problem at hand supposes to obtain asymptotic solutions which can be mathematically devised differently. Although all such solutions describe the motion in the same manner, some conclusions can be made to depend on the structures of the equations obtained [6–7]. In this work it is shown that adopting the method presented we obtain a formula, which reflects not

only the angle of precession but also the shortening of the major axis of the orbit, in comparison with calculations based on Newtonian mechanics. For massive stars such shortening would be of greater importance for the existence of planet systems.

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## ИЗСЛЕДВАНЕ ДВИЖЕНИЕТО НА СПЪТНИК, СЪГЛАСНО ОБЩАТА ТЕОРИЯ НА ОТНОСИТЕЛНОСТТА

*Костадин Шейретски*

### Резюме

В статията се изследва интеграла на енергията, който се получава в задачата за движение на материална точка в централно поле спрямо Обща теория на относителността. Приложени са метод на малкия параметър в комбинация с метод на хармоничния баланс. Изведена е компактна формула, описваща траекторията на движение. Формулата дава правилно количествено описание на основните релативистки ефекти. Доказано е скъсяване на главната полуос на отрбитата, в сравнение със случая когато не се отчитат релативистките ефекти. Този резултат може да бъде полезен при анализиране структурата на планетни системи, образувани около масивни звезди.