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ANALYTICAL COMPUTATION OF TWO INTEGRALS, APPEARING IN THE THEORY OF ELLIPTICAL ACCRETION DISCS. IV. SOLVING OF THE INTEGRALS, ENSURING THE EVALUATION OF THE DERIVATIVES, ENTERING INTO THE WRONSKI DETERMINANT

Dimitar Dimitrov

Space Research and Technology Institute – Bulgarian Academy of Sciences
e-mail: dim@mail.space.bas.bg

Abstract

The present paper deals with the analytical evaluation of the definite integral $\int_0^{2\pi}(1 + e\cos\varphi)^{n-4}[1 + (e - \dot{e})\cos\varphi]^{-n-1}\,d\varphi$, where *e(u) are the eccentricities of the particle orbits,* $\dot{e}(u) \equiv de(u)/du$, $u \equiv ln(p)$, *with **p** being the focal parameter of the corresponding elliptical particle orbits. The parameter **n** is the power in the viscosity law* $\eta = \beta\,\Sigma^n$, *where* $\boldsymbol{\Sigma}$ *is the surface density of the accretion disc, and* $\boldsymbol{\varphi}$ *is the azimuthal angle. We have fulfilled computations under the following three restrictions: (i)* $|e(u)| < 1$, *(ii)* $|\dot{e}(u)| < 1$ *and (iii)* $|e(u) - \dot{e}(u)| < 1$. *They are physically motivated by the accepted for our considerations model of stationary elliptical accretion discs of Lyubarskij et al. [1]. Many particular cases, arising from the singular behavior of some terms for given values of* $e(u)$, $\dot{e}(u)$, *their difference* $e(u) - \dot{e}(u)$ *and the power **n**, are computed in details. These calculations are performed in two ways: (i) by a direct substitution of the singular value into the initial definition of the integral, and (ii) by a limit transition to this singular value into the already evaluated analytical expression for the integral, obtained for the regular values of the corresponding variables. In the later case, the application of the L'Hospital's rule for resolving of indeterminacies of the type 0/0 is very useful. Both the approaches give the same results in every verified case, which ensures that the transition through the singular value is continuous. This means that the analytical solutions for all the considered (singular and non-singular) cases may be combined into one single formula. Such a prescription of the solution of the above written integral is very suitable to the occasion, when this formula is applied for the verification of the linear dependence/independence of the coefficients, entering into the terms of the dynamical equation of the elliptical accretion disc.*

1. Introduction: Some definitions and notations

In the present paper we continue the investigation of the *stationary* elliptical accretion discs, according to the model, developed by Lyubarskij et al. [1]. For more clarity, we shall write down the definitions of the following seven integrals ([2] and the references therein):

(1) $$\mathbf{I_{0-}}(e,\dot{e},n) \equiv \int_0^{2\pi} (1 + e\cos\varphi)^{n-3}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi\ ,$$

(2) $$\mathbf{I_{0+}}(e,\dot{e},n) \equiv \int_0^{2\pi} (1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+2)}\, d\varphi\ ,$$

(3) $$\mathbf{I_j}(e,\dot{e},n) \equiv \int_0^{2\pi} (\cos\varphi)^{\mathrm{j}}(1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi\ ;\ \ \mathbf{j} = 0, 1, 2, 3, 4.$$

The appearance of the above written integrals $\mathbf{I_{0-}}(e,\dot{e},n)$, $\mathbf{I_{0+}}(e,\dot{e},n)$ and $\mathbf{I_j}(e,\dot{e},n)$, ($\mathbf{j}$ = 0, 1, 2, 3, 4), is evident from the previous considerations (and derivations) of the dynamical equation ([1], [2] and the references therein). For this reason, we shall not discuss now such a subject. We mention only that the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$, $\mathbf{I_{0+}}(e,\dot{e},n)$ and $\mathbf{I_j}(e,\dot{e},n)$, ($\mathbf{j}$ = 0, 1, 2, 3, 4) arise, due to the angle-averaging over the azimuthal angle φ in the used system of non-orthogonal curvilinear coordinates (p, φ). Here p is the focal parameter of the elliptical orbit for each particle, which changes for the different parts of the accretion disc. Further in the our exposition, like in the paper [1], instead of p we use its logarithm $u \equiv ln(p)$. Therefore, the eccentricities e of the particle orbits and their derivatives $\dot{e} \equiv de/du$ are functions of the parameter/coordinate u. It is worth to note, that for *circular* orbits in the accretion flow (like the standard α-disc model [3]), the parameter p is simply the radius r of the corresponding particle orbit. To underline that u is an independent coordinate, we shall often write further that $e = e(u)$ and $\dot{e} = \dot{e}(u)$. It also remains to remark that the parameter n is the power into the viscosity law $\eta = \beta\Sigma^{\,n}$, where η is the viscosity, Σ is the surface density of the accretion disc, and β is a constant. We stress that n is a constant throughout the disc, i.e., n does not depend on u. Of course, under the transition from one model of accretion flow to another one, the power n may change from a given value to another constant meaning.

During the process of realization of the our program for simplification of the dynamical equation (derived initially by Lyubarskij et al. [1]; *stationary* case), we strike with the problem of the analytical evaluation of the derivatives with respect to $e(u)$ and $\dot{e}(u)$ of the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$. It can be shown that the first partial derivatives

$\partial\mathbf{I}_{0-}(e,\dot{e},n)/\partial e$, $\partial\mathbf{I}_{0-}(e,\dot{e},n)/\partial\dot{e}$, $\partial\mathbf{I}_{0+}(e,\dot{e},n)/\partial e$ and $\partial\mathbf{I}_{0+}(e,\dot{e},n)/\partial\dot{e}$ may be expressed as linear combinations of the integrals $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$. The exact analytical forms of the coefficients of these linear dependences will be derived in a forthcoming paper [4]. We remind that we have accepted the following approach. Until now we do not know the searched for solution $e(u)$ of the dynamical equation and, consequently, the analytical form of its derivative $\dot{e}(u) \equiv de(u)/du$ is also unknown. Hence, we are able to consider the eccentricity $e(u)$ and its derivative $\dot{e}(u)$ as **"**independent**"** variables, having, however, in mind that under differentiation with respect to u, we must take into account that $\dot{e}(u) \equiv$ $\equiv de(u)/du$. As we have mentioned earlier, ([2] and the references therein), we insert the following three restrictions: (*i*) $|e(u)| < 1$, (*ii*) $|\dot{e}(u)| < 1$ and (*iii*) $|e(u) - \dot{e}(u)| < 1$ for all values of the parameter $u \equiv ln(p)$ (i.e., in fact, for all admissible values of the focal parameter p). Our current problem, which we intend to solve, is the question whether the integrals $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$ **are** linearly independent functions with respect to their arguments $e(u)$, $\dot{e}(u)$ and n, or not ? The standard way to check this is to compute the corresponding *Wronski* determinants. The identical equality to zero of these determinants are *necessary conditions* to be fulfilled the linear relations between the integrals $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$. *If the opposite is true*, then $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$ must be linearly independent functions of $e(u)$, $\dot{e}(u)$ and n, because the pointed out *necessary conditions* would be violated. The analytical evaluations of the ***Wronski*** determinants require computations of second order partial derivatives like $\partial^2\mathbf{I}_{0-}(e,\dot{e},n)/\partial e^2$, $\partial^2\mathbf{I}_{0-}(e,\dot{e},n)/\partial e\partial\dot{e}$, $\partial^2\mathbf{I}_{0-}(e,\dot{e},n)/\partial\dot{e}^2$, $\partial^2\mathbf{I}_{0+}(e,\dot{e},n)/\partial e^2$, $\partial^2\mathbf{I}_{0+}(e,\dot{e},n)/\partial e\partial\dot{e}$ and $\partial^2\mathbf{I}_{0+}(e,\dot{e},n)/\partial\dot{e}^2$. In view of their analytical evaluation, it is appropriate to compute preliminary two auxiliary integrals, defined by the equalities (4) and (5) below in the next chapter 2.

2. Computation of two auxiliary integrals

In our preparation to find explicit analytical expressions for the second order partial derivatives $\partial^2\mathbf{I}_{0-}(e,\dot{e},n)/\partial e^2$, $\partial^2\mathbf{I}_{0-}(e,\dot{e},n)/\partial e\partial\dot{e}$, $\partial^2\mathbf{I}_{0-}(e,\dot{e},n)/\partial\dot{e}^2$, $\partial^2\mathbf{I}_{0+}(e,\dot{e},n)/\partial e^2$, $\partial^2\mathbf{I}_{0+}(e,\dot{e},n)/\partial e\partial\dot{e}$ and $\partial^2\mathbf{I}_{0+}(e,\dot{e},n)/\partial\dot{e}^2$, we encounter with the necessity to evaluate two integrals, namely:

$$(4) \qquad \mathbf{I}_{0,-4,+1}(e,\dot{e},n) \equiv \int_0^{2\pi}(1 + ecos\varphi)^{n-4}[1 + (e - \dot{e})cos\varphi]^{-n-1}\, d\varphi$$

and

(5) $\quad \mathbf{I_{0,-2,+3}}(e,\dot{e},n) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-n-3}\, d\varphi.$

We do not give here a precise designation of the above two integrals. The idea for such (*unconventional*) index notations is that the integrands of the integrals, with which we are dealing, may be represented as a product of three multipliers. Two of them are into the nominator: $(\cos\varphi)^{\textit{first index}}$ and $(1 + e\cos\varphi)^{n\,+\,\textit{second index}}$. The third multiplier is, in fact, the denominator of the integrand: $[1 + (e - \dot{e})\cos\varphi]^{n\,+\,\textit{third index}}$, or, transforming it like a multiplier of the nominator: $[1 + (e - \dot{e})\cos\varphi]^{-(n\,+\,\textit{third index})}$. In particular, if the *first index* = 0, this means that the multiplier $\cos\varphi$ is absent into the nominator. For example, we would be able to write $\mathbf{I_{0-}}(e,\dot{e},n)$ as $\mathbf{I_{0,-3,+1}}(e,\dot{e},n)$, or to write $\mathbf{I_{0+}}(e,\dot{e},n)$ as $\mathbf{I_{0,-2,+2}}(e,\dot{e},n)$. But we shall not change the "old" system of notations. The reason for this is that the integrals (4) and (5) *only temporarily emerge* into our computations and they must not be considered as frequently struck functions in the evaluated formulas. We also underline that by the term "analytical evaluation of the integrals $\mathbf{I_{0,-4,+1}}(e,\dot{e},n)$ and $\mathbf{I_{0,-2,+3}}(e,\dot{e},n)$" we do not understand by all means that the evaluation is finished up to some more or less analytical expressions. Instead of that, it may happen to satisfy us with the more modest conclusion that $\mathbf{I_{0,-4,+1}}(e,\dot{e},n)$ and $\mathbf{I_{0,-2,+3}}(e,\dot{e},n)$ are linear combinations of the integrals (1) – (3). The establishing of such linear relations is fully sufficient for our purposes.

2.1. Evaluation of the integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{n-4}[1 + (e - \dot{e})\cos\varphi]^{-n-1}\, d\varphi$

2.1.1. <u>Case $n \neq 3$, $e(u) \neq 0$, $\dot{e}(u) \neq 0$</u>

According to the definition (4), we perform the following transformations of the considered integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n)$:

(6)
$$\mathbf{I_{0,-4,+1}}(e,\dot{e},n) = \int_0^{2\pi}(1 + e\cos\varphi)^{n-4}(\cos^2\varphi + \sin^2\varphi)[1 + (e - \dot{e})\cos\varphi]^{-n-1}\, d\varphi =$$

$$= - e^{-2}\int_0^{2\pi}(1 + e\cos\varphi)^{n-3}(1 - e\cos\varphi)[1 + (e - \dot{e})\cos\varphi]^{-n-1}\, d\varphi + e^{-2}\int_0^{2\pi}(1 + e\cos\varphi)^{n-4}\times -$$

$$\times[1 + (e - \dot{e})\cos\varphi]^{-n-1}\, d\varphi - [(n-3)e]^{-1}\int_0^{2\pi}(\sin\varphi)[1 + (e - \dot{e})\cos\varphi]^{-n-1}d[(1 + e\cos\varphi)^{n-3}] =$$

$$= e^{-2}\mathbf{I_{0,-4,+1}}(e,\dot{e},n) - e^{-2}\mathbf{I_{0-}}(e,\dot{e},n) + e^{-2}\int_0^{2\pi}(1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-n-1}\, d\varphi -$$

$$- e^{-2}\int_0^{2\pi}(1 + e\cos\varphi)^{n-3}[1 + (e - \dot{e})\cos\varphi]^{-n-1}\, d\varphi + [(n-3)e^2]^{-1}\int_0^{2\pi}(1 + e\cos\varphi)^{n-3}\times$$

$$\times[(1 + e\cos\varphi) - 1][1 + (e - \dot{e})\cos\varphi]^{-n-1}\, d\varphi +$$

$$+ (n+1)(e - \dot{e})[(n-3)e]^{-1}\int_0^{2\pi}(1 + e\cos\varphi)^{n-3}(\sin^2\varphi)[1 + (e - \dot{e})\cos\varphi]^{-n-2}\, d\varphi.$$

Taking into account in the last integral that $\sin^2\varphi = 1 - \cos^2\varphi$, we arrive at the next relation:

(7) $$(1 - e^{-2})\mathbf{I}_{0,-4,+1}(e,\dot{e},n) = - 2e^{-2}\mathbf{I}_{0-}(e,\dot{e},n) + e^{-2}\mathbf{I}_{0}(e,\dot{e},n) + [(n-3)e^2]^{-1}\mathbf{I}_{0}(e,\dot{e},n) -$$

$$- [(n-3)e^2]^{-1}\mathbf{I}_{0-}(e,\dot{e},n) + (n+1)(e - \dot{e})[(n-3)e]^{-1}\int_0^{2\pi}(1 + e\cos\varphi)^{n-3}\times [1 + (e - \dot{e})\cos\varphi]^{-n-2}\, d\varphi +$$

$$+ (n+1)(e - \dot{e})[(n-3)e^3]^{-1}\int_0^{2\pi}(1 + e\cos\varphi)^{n-3}[(1 - e^2\cos^2\varphi) - 1][1 + (e - \dot{e})\cos\varphi]^{-n-2}\, d\varphi.$$

To proceed further, we have to compute the before the last integral in the above equality, namely:

(8) $$\mathbf{I}_{0,-3,+2}(e,\dot{e},n) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{n-3}[1 + (e - \dot{e})\cos\varphi]^{-n-2}\, d\varphi = \int_0^{2\pi}(1 + e\cos\varphi)^{n-3}[1 + (e - \dot{e})\cos\varphi]^{-n-1}\, d\varphi -$$

$$- [(e - \dot{e})/e]\int_0^{2\pi}(1 + e\cos\varphi)^{n-3}[(1 + e\cos\varphi) - 1][1 + (e - \dot{e})\cos\varphi]^{-n-2}\, d\varphi =$$

$$= \mathbf{I}_{0-}(e,\dot{e},n) - [(e - \dot{e})/e]\mathbf{I}_{0+}(e,\dot{e},n) + [(e - \dot{e})/e]\mathbf{I}_{0,-3,+2}(e,\dot{e},n),$$

where we have used the same manner of notations as for the integral $\mathbf{I}_{0,-4,+1}(e,\dot{e},n)$ and $\mathbf{I}_{0,-2,+3}(e,\dot{e},n)$. Consequently, from the relation (8) we have:

(9) $$\mathbf{I}_{0,-3,+2}(e,\dot{e},n) = (e/\dot{e})\mathbf{I}_{0-}(e,\dot{e},n) - [(e - \dot{e})/\dot{e}]\mathbf{I}_{0+}(e,\dot{e},n).$$

We remark that the above relation is derived under the conditions $e(u) \neq 0$ and $\dot{e}(u) \neq 0$, comprised in the Case 2.1.1. Substitution of (9) into (7) gives:

(10) $$(1 - e^{-2})\mathbf{I}_{0,-4,+1}(e,\dot{e},n) = - (2n-5)[(n-3)e^2]^{-1}\mathbf{I}_{0-}(e,\dot{e},n) + (n-2)[(n-3)e^2]^{-1}\mathbf{I}_{0}(e,\dot{e},n) +$$
$$+ (n+1)(e - \dot{e})[(n-3)\dot{e}]^{-1}\mathbf{I}_{0-}(e,\dot{e},n) - (n+1)(e - \dot{e})^2[(n-3)e\dot{e}]^{-1}\mathbf{I}_{0+}(e,\dot{e},n) +$$

$$+ (n+1)(e - \dot{e})[(n-3)e^3]^{-1}\int_0^{2\pi}(1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-n-2}\, d\varphi -$$

$$- (n+1)[(n-3)e^2]^{-1}\int_0^{2\pi}(1 + e\cos\varphi)^{n-2}\{[1 + (e - \dot{e})\cos\varphi] - 1\}[1 + (e - \dot{e})\cos\varphi]^{-n-2}\, d\varphi -$$

$$- (n+1)(e - \dot{e})[(n-3)e^3]^{-1}\mathbf{I}_{0,-3,+2}(e,\dot{e},n).$$

Applying again the result (9), we obtain that:

(11) $$(1 - e^{-2})\mathbf{I}_{0,-4,+1}(e,\dot{e},n) = (n-3)^{-1}[(-2n+5)/e^2 + (n+1)(e - \dot{e})/\dot{e} - (n+1)(e - \dot{e})(e^2\dot{e})^{-1}]\mathbf{I}_{0-}(e,\dot{e},n) +$$
$$+ (n-3)^{-1}[- (n+1)(e - \dot{e})^2(e\dot{e})^{-1} + (n+1)(e - \dot{e})e^{-3} + (n+1)e^{-2} + (n+1)(e - \dot{e})^2(e^3\dot{e})]\mathbf{I}_{0+}(e,\dot{e},n) +$$
$$+ (n-3)^{-1}[(n-2)e^{-2} - (n+1)e^{-2}]\mathbf{I}_{0}(e,\dot{e},n).$$

Multiplying the above equality by $[-(n-3)e^2]$, we obtain:

(12) $$(n-3)(1 - e^2)\mathbf{I}_{0,-4,+1}(e,\dot{e},n) = [(2n-5) + (n+1)(1 - e^2)(e - \dot{e})\dot{e}^{-1}]\mathbf{I}_{0-}(e,\dot{e},n) -$$
$$- (n+1)[(1 - e^2)(e - \dot{e})^2(e\dot{e})^{-1} + (2e - \dot{e})/e]\mathbf{I}_{0+}(e,\dot{e},n) + 3\mathbf{I}_{0}(e,\dot{e},n).$$

This is the wanted representation of the integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n)$ through the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$, $\mathbf{I_{0+}}(e,\dot{e},n)$ and $\mathbf{I_0}(e,\dot{e},n)$. Obviously, this dependence is linear and is derived under the conditions $(n - 3) \neq 0$, $e(u) \neq 0$ and $\dot{e}(u) \neq 0$. No matter if $e(u) - \dot{e}(u) \neq 0$ or $e(u) - \dot{e}(u) = 0$!

Before to proceed further, we shall remark that we have already computed the analytical expressions for the integrals

$\mathbf{A_i}(e,\dot{e}) \equiv \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-\mathrm{i}}\, d\varphi$; ($\mathbf{i}$ = 1, 2, 3, 4, 5),

$\mathbf{J_i}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-\mathrm{i}}\, d\varphi$; ($\mathbf{i}$ = 1, 2, 3, 4), $\mathbf{H_i}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + ecos\varphi)^{-\mathrm{i}} \times$ $\times\ [1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi$; ($\mathbf{i}$ = 1, 2, 3, 4).Their detailed evaluations are carried out in paper [5].

We also have analytical estimations for the integrals $\mathbf{L_i}(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + ecos\varphi)](1 +$ $+\ ecos\varphi)^{-1}\times$ $\times[1 + (e - \dot{e})cos\varphi]^{-\mathrm{i}}\, d\varphi$; ($\mathbf{i}$ = 0, 1, 2, 3), $\mathbf{K_i}(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-\mathrm{i}}\, d\varphi$; ($\mathbf{i}$ = 1, 2, 3, 4, 5), which derivations are circumstantially described in the papers [6] and [7]. We shall often quote these results, in order to argue our further calculations. Also we shall take into use the expressions of the above integrals for some particular values of their arguments $e(u)$ and $\dot{e}(u)$, which are cited in the above mentioned papers [5], [6] and [7]. *In fact, the later three works were preliminary worked out, in view of their application to the needs of the present paper, i.e., they are in that sense, auxiliary investigations*.

2.1.2. Case $n \neq 3$, $e(u) \neq 0$, $\dot{e}(u) = 0$ => $e(u) - \dot{e}(u) \neq 0$

According to the definitions (1) – (3), we can write for $\dot{e}(u) = 0$ the following expressions for the integrals $\mathbf{I_0}(e,\dot{e} = 0,n)$, $\mathbf{I_{0-}}(e,\dot{e} = 0,n)$, $\mathbf{I_{0+}}(e,\dot{e} = 0,n)$ and $\mathbf{I_{0,-4,+1}}(e,\dot{e} = 0,n)$:

(13) $\mathbf{I_0}(e,\dot{e} = 0,n) \equiv \int_0^{2\pi}(1 + ecos\varphi)^{-3}\, d\varphi \equiv \mathbf{A_3}(e, 0) = \pi(2 + e^2)(1 - e^2)^{-5/2}$, (eq. (22) from paper [5]),

(14) $\mathbf{I_{0-}}(e,\dot{e} = 0,n) = \mathbf{I_{0+}}(e,\dot{e} = 0,n) \equiv \int_0^{2\pi}(1 + ecos\varphi)^{-4}\, d\varphi \equiv \mathbf{A_4}(e, 0) = \pi(2 + 3e^2)(1 - e^2)^{-7/2}$,

(eq. (23) from paper [5]), and

(15) $$\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n) \equiv \int_0^{2\pi}(1 + ecos\varphi)^{-5}\, d\varphi \equiv \mathbf{A}_\mathbf{5}(e, 0) = (\pi/4)(8 + 24e^2 + 3e^4)(1 - e^2)^{-9/2},$$

(eq. (24) from paper [5]).

Note that the above three evaluations (13) – (15) do not depend on the power n in the viscosity law $\eta = \beta\Sigma^n$. They are valid also for $n = 3$! We can perform the following transformation of the relations (14):

(16) $$\mathbf{I}_{\mathbf{0-}}(e,\dot{e} = 0,n) = \mathbf{I}_{\mathbf{0+}}(e,\dot{e} = 0,n) = \int_0^{2\pi}(1 + ecos\varphi)^{-3}\, d\varphi - e\int_0^{2\pi}(1 + ecos\varphi)^{-4}\, d(sin\varphi) = \mathbf{I}_\mathbf{0}(e,\dot{e} = 0,n) +$$

$$+ 4e^2\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n) - 4\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n) + 4\int_0^{2\pi}(1 + ecos\varphi)(1 - ecos\varphi)(1 + ecos\varphi)^{-5}\, d\varphi =$$

$$= \mathbf{I}_\mathbf{0}(e,\dot{e} = 0,n) + 4(e^2 - 1)\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n) + 4\mathbf{I}_{\mathbf{0-}}(e,\dot{e} = 0,n) - 4\int_0^{2\pi}(1 + ecos\varphi)^{-3}\, d\varphi +$$

$$+ 4\int_0^{2\pi}(1 + ecos\varphi)^{-4}\, d\varphi = \mathbf{I}_\mathbf{0}(e,\dot{e} = 0,n) + 4(e^2 - 1)\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n) + 4\mathbf{I}_{\mathbf{0-}}(e,\dot{e} = 0,n) - 4\mathbf{I}_\mathbf{0}(e,\dot{e} = 0,n) +$$

$$+ 4\mathbf{I}_{\mathbf{0-}}(e,\dot{e} = 0,n) = 8\mathbf{I}_{\mathbf{0-}}(e,\dot{e} = 0,n) - 3\mathbf{I}_\mathbf{0}(e,\dot{e} = 0,n) + 4(e^2 - 1)\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n).$$

From this equality we are able to express the integral $\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n)$ through the integrals $\mathbf{I}_{\mathbf{0-}}(e,\dot{e} = 0,n) = \mathbf{I}_{\mathbf{0+}}(e,\dot{e} = 0,n)$ and $\mathbf{I}_\mathbf{0}(e,\dot{e} = 0,n)$. Consequently, dividing by $4(1 - e^2) \neq 0$, we obtain:

(17) $$\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n) = [4(1 - e^2)]^{-1}[7\mathbf{I}_{\mathbf{0-}}(e,\dot{e} = 0,n) - 3\mathbf{I}_\mathbf{0}(e,\dot{e} = 0,n)].$$

We again note that the above relation (17) is also valid for $n = 3$, because under its deduction we do not require anywhere the condition $n \neq 3$ to be fulfilled. It is evident also that (17) remains valid for $e(u) = 0$. In the later case, the equality (17) can be written as:

(18) $$2\pi = (1/4)[7(2\pi) - 3(2\pi)],$$

which is obviously true.

Let us rewrite the relation (13) in the following way, in order to see its validity under the transition $\dot{e}(u) \to 0$:

(19) $$(n - 3)(1 - e^2)\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e},n) = (2n - 5)\mathbf{I}_{\mathbf{0-}}(e,\dot{e},n) - (n + 1)(2e - \dot{e})e^{-1}\mathbf{I}_{\mathbf{0+}}(e,\dot{e},n) +$$
$$+ 3\mathbf{I}_\mathbf{0}(e,\dot{e},n) + (n + 1)(1 - e^2)(e - \dot{e})\dot{e}^{-1}\{\mathbf{I}_{\mathbf{0-}}(e,\dot{e},n) - [(e - \dot{e})/e]\mathbf{I}_{\mathbf{0+}}(e,\dot{e},n)\}.$$

We see that:

(20) $$\lim_{\dot{e}(u) \to 0}\{\mathbf{I}_{\mathbf{0-}}(e,\dot{e},n) - [(e - \dot{e})/e]\mathbf{I}_{\mathbf{0+}}(e,\dot{e},n)\} = \mathbf{I}_{\mathbf{0-}}(e,\dot{e} = 0,n) - \mathbf{I}_{\mathbf{0+}}(e,\dot{e} = 0,n) = 0,$$

according to the equalities (14). This result ensures that we may apply the L'Hospital's theorem for computing of indeterminacies of the type 0/0. Because $\partial\dot{e}(u)/\partial\dot{e} = 1$, it is enough to evaluate the derivative:

(21) $$\partial\{\mathbf{I}_{\mathbf{0-}}(e,\dot{e},n) - [(e - \dot{e})/e]\mathbf{I}_{\mathbf{0+}}(e,\dot{e},n)\}/\partial\dot{e} = (n + 1)\int_0^{2\pi}(cos\varphi)(1 + ecos\varphi)^{n-3}[1 + (e - \dot{e})cos\varphi]^{-n-2}\, d\varphi +$$

$$+ e^{-1}\int_0^{2\pi}(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n-2}\, d\varphi - (n + 2)[(e - \dot{e})/e]\int_0^{2\pi}(cos\varphi)(1 + ecos\varphi)^{n-2}\times$$

$$\times[1 + (e - \dot{e})cos\varphi]^{-n-3}\, d\varphi \xrightarrow[\dot{e}(u) \to 0]{} (n + 1)\int_0^{2\pi}(cos\varphi)(1 + ecos\varphi)^{n-3}(1 + ecos\varphi)^{-n-2}\, d\varphi +$$

$$+ e^{-1}\int_0^{2\pi}(1 + ecos\varphi)^{n-2}(1 + ecos\varphi)^{-n-2}\, d\varphi - (n + 2)\int_0^{2\pi}(cos\varphi)(1 + ecos\varphi)^{n-2}(1 + ecos\varphi)^{-n-3}\, d\varphi =$$

$$- e^{-1}\int_0^{2\pi}(1 + ecos\varphi)(1 + ecos\varphi)^{-5}\, d\varphi + e^{-1}\int_0^{2\pi}(1 + ecos\varphi)^{-5}\, d\varphi + e^{-1}\int_0^{2\pi}(1 + ecos\varphi)^{-4}\, d\varphi =$$

$$= e^{-1}\int_0^{2\pi}(1 + ecos\varphi)^{-5}\, d\varphi = e^{-1}\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n).$$

Therefore:

$$(22) \qquad \lim_{\dot{e}(u) \to 0}\{\dot{e}^{-1}\{\mathbf{I}_{\mathbf{0}-}(e,\dot{e},n) - [(e - \dot{e})/e]\mathbf{I}_{\mathbf{0}+}(e,\dot{e},n)\}\} = e^{-1}\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n).$$

So that, we have computed the problematic multiplier, associated with the transition $\dot{e}(u) \to 0$. Now, we take this limit $\dot{e}(u) \to 0$ for the whole relation (19), taking into account that $\mathbf{I}_{\mathbf{0}+}(e,\dot{e} = 0,n) = \mathbf{I}_{\mathbf{0}-}(e,\dot{e} = 0,n)$ (see the equality (14)):

$$(23) \qquad 4(1 - e^2)\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n) = 7\mathbf{I}_{\mathbf{0}-}(e,\dot{e} = 0,n) - 3\mathbf{I}_{\mathbf{0}}(e,\dot{e} = 0,n).$$

After dividing by $4(1 - e^2) \neq 0$, we obtain the relation (17). Consequently, we may consider the equality (12) as valid also for the case $\dot{e}(u) = 0$, keeping in mind that we have to perform the limit transition $\dot{e}(u) \to 0$ with the help of the L'Hospital's theorem for evaluation of uncertainties of the type 0/0. We again stress that the above results do not use the restriction $(n - 3) \neq 0$, and may be applied to the case $e(u) = 0$.

2.1.3. Case $n \neq 3$, $e(u) = 0$, $\dot{e}(u) \neq 0$ => $e(u) - \dot{e}(u) \neq 0$

Taking into account the definitions (1) – (4), we can write for the present case the following equalities, as concerns to the integrals $\mathbf{I}_{\mathbf{0}}(e = 0,\dot{e},n)$, $\mathbf{I}_{\mathbf{0}-}(e = 0,\dot{e},n)$, $\mathbf{I}_{\mathbf{0}+}(e = 0,\dot{e},n)$ and $\mathbf{I}_{\mathbf{0,-4,+1}}(e = 0,\dot{e},n)$:

$$(24) \qquad \mathbf{I}_{\mathbf{0}-}(e = 0,\dot{e},n) = \mathbf{I}_{\mathbf{0}}(e = 0,\dot{e},n) = \mathbf{I}_{\mathbf{0,-4,+1}}(e = 0,\dot{e},n) \equiv \int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-n-1}\, d\varphi,$$

$$(25) \qquad \mathbf{I}_{\mathbf{0}+}(e = 0,\dot{e},n) \equiv \int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-n-2}\, d\varphi > 0.$$

Note that the equalities (24) are valid also for $n = 3$ and also for $\dot{e}(u) = 0$, when $\mathbf{I}_{\mathbf{0,-4,+1}}(e = 0,\dot{e} = 0,n) = 2\pi$. Let us rewrite the relation (12) in the following way, in order to see its validity under the limit transition

$e(u) \to 0$. That is to say, we detach the multiplier, which contains into its denominator the factor $e(u)$, causing the troubles under this transition $e(u) \to 0$:

(26) $$(n-3)(1-e^2)\mathbf{I}_{\mathbf{0},-4,+1}(e,\dot{e},n) = [(2n-5)+(n+1)(1-e^2)(e-\dot{e})\dot{e}^{-1}]\mathbf{I}_{\mathbf{0}-}(e,\dot{e},n) - $$
$$-(n+1)e^{-1}[(1-e^2)(e-\dot{e})^2\dot{e}^{-1}+2e-\dot{e}]\mathbf{I}_{\mathbf{0}+}(e,\dot{e},n)+3\mathbf{I}_{\mathbf{0}}(e,\dot{e},n).$$

The problematic term is the coefficient of the integral $\mathbf{I}_{\mathbf{0}+}(e,\dot{e},n)$ in the right-hand-side of (26), which contains into its denominator the multiplier $e(u)$. We see that:

(27) $$\lim_{e(u)\to 0}[(1-e^2)(e-\dot{e})^2\dot{e}^{-1}+2e-\dot{e}] = \dot{e}^2\dot{e}^{-1}-\dot{e} = 0.$$

Further, we compute the derivative:

(28) $$\partial[(1-e^2)(e-\dot{e})^2\dot{e}^{-1}+2e-\dot{e}]/\partial e = \dot{e}^{-1}[-2e(e-\dot{e})^2+2(1-e^2)(e-\dot{e})]+2 \xrightarrow[e(u)\to 0]{} -2\dot{e}/\dot{e}+2 = 0.$$

We may apply again the L'Hospital's theorem to obtain that:

(29) $$\lim_{e(u)\to 0}\{e^{-1}[(1-e^2)(e-\dot{e})^2\dot{e}^{-1}+2e-\dot{e}]\} = 0.$$

This means that, if we take the limit $e(u) \to 0$ for the both sides of the equation (26), the coefficient before the integral $\mathbf{I}_{\mathbf{0}+}(e,\dot{e},n)$ will become equal to zero:

(30) $$(n-3)\mathbf{I}_{\mathbf{0},-4,+1}(e=0,\dot{e},n) = (n-6)\mathbf{I}_{\mathbf{0}-}(e=0,\dot{e},n)+3\mathbf{I}_{\mathbf{0}}(e=0,\dot{e},n),$$

or, with the reading of the first equality in (24):

(31) $$(n-3)\mathbf{I}_{\mathbf{0},-4,+1}(e=0,\dot{e},n) = (n-3)\mathbf{I}_{\mathbf{0}-}(e=0,\dot{e},n).$$

Taking into account that in the presently considered case $n \neq 3$, we may cancel out the factor $(n-3)$ and to obtain the second equality in the relation (24). As we already mentioned above, it is also valid if we set into it $\dot{e}(u) = 0$:

(32) $$\mathbf{I}_{\mathbf{0},-4,+1}(e=0,\dot{e}=0,n) = \mathbf{I}_{\mathbf{0}-}(e=0,\dot{e}=0,n) = 2\pi.$$

Therefore, under the limit transition $e(u) \to 0$, the relation (12) leads to the right equality (31). Consequently, we are able to consider (12) to remain valid also for $e(u) = 0$, having in mind that then we must apply the L'Hospital's theorem for revealing of uncertainties of the type 0/0.

2.1.4. Case $n \neq 3$, $e(u) = 0$, $\dot{e}(u) = 0$ => $e(u) - \dot{e}(u) = 0$

It is easily seen that (both for $n \neq 3$ and $n = 3$):

(33) $$\mathbf{I}_{\mathbf{0},-4,+1}(e=0,\dot{e}=0,n) = \mathbf{I}_{\mathbf{0}-}(e=0,\dot{e}=0,n) = \mathbf{I}_{\mathbf{0}}(e=0,\dot{e}=0,n) = \mathbf{I}_{\mathbf{0}+}(e=0,\dot{e}=0,n) = 2\pi.$$

Taking into account the correctness of the above equalities, we may apply the analytical representation (12) also in the present case, after performing the transitions

$[e(u) \to 0] \cap [\dot{e}(u) \to 0]$ or $[\dot{e}(u) \to 0] \cap [e(u) \to 0]$. No matter which of the transitions is taken in the first place!

2.1.5.1. Case $n = 3$, $e(u) \neq 0$, $\dot{e}(u) \neq 0$, $e(u) - \dot{e}(u) \neq 0$

2.1.5.1.1. A direct computation of the integral $\mathbf{I}_{0,-4,+1}(e,\dot{e},n = 3)$ through the integral $\mathbf{I}_{0-}(e,\dot{e},n = 3)$

Substituting $n = 3$ into the definitions (1) – (4), we shall have the following representations for the integrals $\mathbf{I}_{0}(e,\dot{e},n = 3)$, $\mathbf{I}_{0-}(e,\dot{e},n = 3)$, $\mathbf{I}_{0+}(e,\dot{e},n = 3)$ and $\mathbf{I}_{0,-4,+1}(e,\dot{e},n = 3)$:

(34) $\mathbf{I}_{0-}(e,\dot{e},n = 3) = \mathbf{A}_4(e,\dot{e}) = \pi[2 + 3(e - \dot{e})^2][1 - (e - \dot{e})^2]^{-7/2}$,

(formula (9) from paper [5]),

(35) $\mathbf{I}_{0+}(e,\dot{e},n = 3) \equiv \int_0^{2\pi}(1 + e\cos\varphi)[1 + (e - \dot{e})\cos\varphi]^{-5}\, d\varphi,$

(36) $\mathbf{I}_{0}(e,\dot{e},n = 3) \equiv \int_0^{2\pi}(1 + e\cos\varphi)[1 + (e - \dot{e})\cos\varphi]^{-4}\, d\varphi,$

(37) $\mathbf{I}_{0,-4,+1}(e,\dot{e},n = 3) = \mathbf{J}_4(e,\dot{e}) =$
$= \pi(-2e^4 + 6e^6 - 6e^8 + 2e^{10} - 14e^5\dot{e} + 28e^7\dot{e} - 14e^9\dot{e} + 7e^4\dot{e}^2 - 49e^6\dot{e}^2 + 42e^8\dot{e}^2 + 35e^5\dot{e}^3 - 70e^7\dot{e}^3 +$
$+ 2\dot{e}^4 + 8e^2\dot{e}^4 + 70e^6\dot{e}^4 - 10e\dot{e}^5 - 14e^3\dot{e}^5 - 42e^5\dot{e}^5 + 3\dot{e}^6 + 7e^2\dot{e}^6 + 14e^4\dot{e}^6 - e\dot{e}^7 - 2e^3\dot{e}^7) +$
$+ 2\pi e^4\dot{e}^{-4}(1 - e^2)^{-1/2}$,

(see formula (47) from paper [5] for the explicit writing of the integral $\mathbf{J}_4(e,\dot{e})$ as a function of the variables $e(u)$ and $\dot{e}(u) \equiv de(u)/du$).

For the purposes, which will become evident from the consequent exposition of the text, we shall not use directly the above written solution for the integral $\mathbf{I}_{0,-4,+1}(e,\dot{e},n = 3)$ (37). Instead of that, we begin with a transformation of the first equality in (37), in order to introduce into the right-hand-side of (37) the integral $\mathbf{I}_{0-}(e,\dot{e},n = 3)$. We intend further to express the integrals $\mathbf{I}_{0+}(e,\dot{e},n = 3)$ and $\mathbf{I}_{0}(e,\dot{e},n = 3)$ (for which we do not give until now any explicit solutions in the formulas (35) and (36), respectively) through the later integral $\mathbf{I}_{0-}(e,\dot{e},n = 3)$.

(38) $\mathbf{I}_{0,-4,+1}(e,\dot{e},n = 3) = \int_0^{2\pi}[(1 + e\cos\varphi) - e\cos\varphi](1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-4}\, d\varphi = \mathbf{I}_{0-}(e,\dot{e},n = 3) -$

$- [e/(e - \dot{e})]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-3}\, d\varphi + [e/(e - \dot{e})]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-4}\, d\varphi =$

$= \mathbf{I}_{0-}(e,\dot{e},n = 3) - [e/(e - \dot{e})]\mathbf{J}_3(e,\dot{e}) + [e/(e - \dot{e})]\mathbf{I}_{0,-4,+1}(e,\dot{e},n = 3),$

where we have used the definitions (34) for $\mathbf{I}_{0-}(e,\dot{e},n = 3) \equiv \mathbf{A}_4(e,\dot{e})$, (37) for $\mathbf{I}_{0,-4,+1}(e,\dot{e},n = 3)$ and (40) from paper [5] for the integral $\mathbf{J}_3(e,\dot{e})$. Therefore, the above equality (38) gives an expression for the considered integral

$\mathbf{I_{0,-4,+1}}(e,\dot{e},n = 3)$.The division by $[-\dot{e}/(e-\dot{e})] \neq 0$ (this is ensured for the examined **Case 2.1.5.1**), gives that:

(39) $\mathbf{I_{0,-4,+1}}(e,\dot{e},n = 3) = \mathbf{J_4}(e,\dot{e}) = -[(e-\dot{e})/\dot{e}]\mathbf{I_{0-}}(e,\dot{e},n = 3) + (e/\dot{e})\mathbf{J_3}(e,\dot{e})$.

In the earlier paper [5], we have derived several recurrent relations with respect to the integrals $\mathbf{J_1}(e,\dot{e})$, $\mathbf{J_2}(e,\dot{e})$ and $\mathbf{J_3}(e,\dot{e})$:

(40) $\mathbf{J_3}(e,\dot{e}) = -[(e-\dot{e})/\dot{e}]\mathbf{A_3}(e,\dot{e}) + (e/\dot{e})\mathbf{J_2}(e,\dot{e})$,

(first equality from the relation (42) in paper [5]; for brevity's sake, we omit the writing out of the explicit dependence of $\mathbf{J_3}(e,\dot{e})$ on $e(u)$ and $\dot{e}(u)$, given by the last equality in (42)),

(41) $\mathbf{J_2}(e,\dot{e}) = -2\pi[(e-\dot{e})/\dot{e}][1-(e-\dot{e})^2]^{-3/2} + (e/\dot{e})\mathbf{J_1}(e,\dot{e})$, (formula (33) from paper [5]).

Of course, we need also of the expression (27) from paper [5], giving the explicit analytical solution for the "initial" integral $\mathbf{J_1}(e,\dot{e})$:

(42) $\mathbf{J_1}(e,\dot{e}) = (2\pi/\dot{e})\{e(1-e^2)^{-1/2} - (e-\dot{e})[1-(e-\dot{e})^2]^{-1/2}\}$.

It remains to replace, in turn, (42) into (41), and after then (41) into (40), in order to eliminate from the equality (40) the integral $\mathbf{J_2}(e,\dot{e})$. But before to make into use this result, intending to resolve for the left-hand-side of the equation (39), we want to derive a presentation of the integral $\mathbf{A_3}(e,\dot{e})$ through the integral $\mathbf{I_{0-}}(e,\dot{e},n = 3)$. Let us compute the auxiliary integral $\mathbf{A_3}(e,\dot{e})$ in the following way:

(43) $$\mathbf{A_3}(e,\dot{e}) = \int_0^{2\pi}[1+(e-\dot{e})cos\varphi]^{-2}\,d\varphi - (e-\dot{e})\int_0^{2\pi}(cos\varphi)[1+(e-\dot{e})cos\varphi]^{-3}\,d\varphi.$$

Earlier we have already found that:

(44) $$\mathbf{A_2}(e,\dot{e}) \equiv \int_0^{2\pi}[1+(e-\dot{e})cos\varphi]^{-2}\,d\varphi = 2\pi[1-(e-\dot{e})^2]^{-3/2},$$

(formula (8) from paper [5]).Therefore:

(45) $$\mathbf{A_3}(e,\dot{e}) = 2\pi[1-(e-\dot{e})^2]^{-3/2} - (e-\dot{e})\int_0^{2\pi}(cos\varphi)[1+(e-\dot{e})cos\varphi]^{-3}\,d\varphi.$$

Developing further the right-hand-side of the above relation (45), we have:

(46) $$\mathbf{A_3}(e,\dot{e}) = 2\pi[1-(e-\dot{e})^2]^{-3/2} - (e-\dot{e})\int_0^{2\pi}[1+(e-\dot{e})cos\varphi]^{-3}\,d(sin\varphi) = 2\pi[1-(e-\dot{e})^2]^{-3/2} +$$
$$+ 3(e-\dot{e})^2\int_0^{2\pi}[1+(e-\dot{e})cos\varphi]^{-4}\,d\varphi + 3\int_0^{2\pi}\{[1-(e-\dot{e})^2cos^2\varphi]-1\}[1+(e-\dot{e})cos\varphi]^{-4}\,d\varphi =$$
$$= 2\pi[1-(e-\dot{e})^2]^{-3/2} + 3(e-\dot{e})^2\mathbf{I_{0-}}(e,\dot{e},n = 3) + 3\int_0^{2\pi}[1+(e-\dot{e})cos\varphi]^{-3}\,d\varphi -$$

$$– 3(e – \dot{e})\int_0^{2\pi}(cos\varphi)[1 + (e – \dot{e})cos\varphi]^{–3}\,d\varphi – 3\mathbf{I_{0-}}(e,\dot{e},n = 3),$$

where we have used the representation (34) for the integral $\mathbf{I_{0-}}(e,\dot{e},n = 3)$, when the power $n = 3$. Consequently, the equalities (46) lead to the result, which can be rewritten as:

(47) $$– 2\mathbf{A_3}(e,\dot{e}) = 2\pi[1 – (e – \dot{e})^2]^{–3/2} + 3[(e – \dot{e})^2 – 1]\mathbf{I_{0-}}(e,\dot{e},n = 3) – 3(e – \dot{e})\int_0^{2\pi}(cos\varphi)\times$$
$$\times[1 + (e – \dot{e})cos\varphi]^{–3}\,d\varphi = – 4\pi[1 – (e – \dot{e})^2]^{–3/2} + 2(e – \dot{e})\int_0^{2\pi}(cos\varphi)[1 + (e – \dot{e})cos\varphi]^{–3}\,d\varphi.$$

The second equality in the right-hand-side of (47) follows from (45) (after the multiplication of (45) by – 2). From (47) we evaluate the integral for which we are looking up. After dividing by – 5, we obtain that:

(48) $$– (e – \dot{e})\int_0^{2\pi}(cos\varphi)[1 + (e – \dot{e})cos\varphi]^{–3}\,d\varphi = – (6\pi/5)[1 – (e – \dot{e})^2]^{–3/2} + (3/5)[1 – (e – \dot{e})^2]\mathbf{I_{0-}}(e,\dot{e},n = 3).$$

Substituting the above result (48) into (45), we arrive at the final expression for the integral $\mathbf{A_3}(e,\dot{e})$:

(49) $$\mathbf{A_3}(e,\dot{e}) = (4\pi/5)[1 – (e – \dot{e})^2]^{–3/2} + (3/5)[1 – (e – \dot{e})^2]\mathbf{I_{0-}}(e,\dot{e},n = 3).$$

Now we are ready to combine the solutions (42), (41) and (49), in order to express the integral $\mathbf{J_3}(e,\dot{e})$ through the integral $\mathbf{I_{0-}}(e,\dot{e},n = 3)$ by means of the recurrence relation (40). At first, from (42) and (41) it may be evaluated that:

(50) $$\mathbf{J_2}(e,\dot{e}) = – 2\pi(e – \dot{e})\dot{e}^{–1}[1 – (e – \dot{e})^2]^{–3/2} – 2\pi e(e – \dot{e})\dot{e}^{–2}[1 – (e – \dot{e})^2]^{–1/2} + 2\pi e^2\dot{e}^{–2}(1 – e^2)^{–1/2}.$$

Then, the relation (40) leads to the expression:

(51) $$\mathbf{J_3}(e,\dot{e}) = – (4\pi/5)(e – \dot{e})\dot{e}^{–1}[1 – (e – \dot{e})^2]^{–3/2} – 2\pi e(e – \dot{e})\dot{e}^{–2}[1 – (e – \dot{e})^2]^{–3/2} –$$
$$– 2\pi e^2(e – \dot{e})\dot{e}^{–3}[1 – (e – \dot{e})^2]^{–1/2} + 2\pi e^3\dot{e}^{–3}(1 – e^2)^{–1/2} – (3/5)(e – \dot{e})[1 – (e – \dot{e})^2]\dot{e}^{–1}\mathbf{I_{0-}}(e,\dot{e},n = 3).$$

Of course, if we replace the analytical expression for the integral $\mathbf{I_{0-}}(e,\dot{e},n = 3) \equiv \mathbf{A_4}(e,\dot{e}) = \pi[2 + 3(e – \dot{e})^2][1 – (e – \dot{e})^2]^{–7/2}$ (see the relation (34) in the present paper) into (51), we have to obtain the representation (42) from the paper [5] for the same integral $\mathbf{J_3}(e,\dot{e})$. We shall not perform here this checking.

Finally, having available the analytical solution for the integral $\mathbf{J_3}(e,\dot{e})$, written into the form (51), we are able to replace it into the equation (39), eliminating thus this integral. Consequently, we conclude that under the conditions, accepted for the considered at present Case 2.1.5.1, the integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n = 3)$, which we are seeking for, takes the following form (here we do not use the explicit solution (34)):

(52) $$\mathbf{I_{0,-4,+1}}(e,\dot{e},n = 3) = \mathbf{J_4}(e,\dot{e}) = – (4\pi/5)e(e – \dot{e})\dot{e}^{–2}[1 – (e – \dot{e})^2]^{–3/2} –$$
$$– 2\pi e^2(e – \dot{e})\dot{e}^{–3}[1 – (e – \dot{e})^2]^{–3/2} – 2\pi e^3(e – \dot{e})\dot{e}^{–4}[1 – (e – \dot{e})^2]^{–1/2} +$$

$+ 2\pi e^4 \dot{e}^{-4}(1 - e^2)^{-1/2} - [(e - \dot{e})/\dot{e}]\{1 + (3/5)e[1 - (e - \dot{e})^2]\dot{e}^{-1}\}\mathbf{I_{0-}}(e,\dot{e},n = 3).$

As before, the remark given above, and concerning the replacement in (52) of the integral $\mathbf{I_{0-}}(e,\dot{e},n = 3)$ with its analytical solution (34), remains valid also for the present situation. Such a substitution of the relation (34) into (52) leads to the analytical expression for $\mathbf{J_4}(e,\dot{e})$, found in the paper [5] (formula (47) in this paper). Again, we shall not check this equivalence of the formulae (52) and (47) in paper [5], because of the brevity reasons.

2.1.5.1.2. Evaluation of the integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n = 3)$ through the limit transition $n \to 3$

We have computed the *explicit* analytical expression (52) for the integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n = 3)$, preserving the existence of the integral $\mathbf{I_{0-}}(e,\dot{e},n =$ $= 3)$, unlike the analytical solution (37). Now we ask: are we able to use the relation (12) (derived under the condition $n \neq 3$) in the limit $n \to 3$, to obtain the solution (52)? The later is computed through *a direct* substitution $n = 3$ into the initial definition (37) for the integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n = 3)$, in accordance with the *general* definition (4) for the integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n)$. For this purpose, we shall try to evaluate the integrals $\mathbf{I_0}(e,\dot{e},n = 3)$ and $\mathbf{I_{0+}}(e,\dot{e},n = 3)$ also by means of the integral $\mathbf{I_{0-}}(e,\dot{e},n = 3)$. This will enable us to check whether the right-hand-side of the equality (12) tends to zero, when n approaches 3, and then to try to apply the L'Hospital's rule for evaluation of indeterminacies of the type 0/0.

Let us write out the integrals $\mathbf{I_0}(e,\dot{e},n = 3)$ and $\mathbf{I_{0+}}(e,\dot{e},n = 3)$ through the integral $\mathbf{I_{0-}}(e,\dot{e},n = 3)$. We have the following expression (see definition (36)):

(53) $$\mathbf{I_0}(e,\dot{e},n = 3) = \int_0^{2\pi}\{[1 + (e - \dot{e})cos\varphi] + \dot{e}cos\varphi\}[1 + (e - \dot{e})cos\varphi]^{-4}\, d\varphi = \mathbf{A_3}(e,\dot{e}) +$$

$$+ (\dot{e}/e)\int_0^{2\pi}[(1 + ecos\varphi) - 1][1 + (e - \dot{e})cos\varphi]^{-4}\, d\varphi = \mathbf{A_3}(e,\dot{e}) + (\dot{e}/e)\mathbf{I_0}(e,\dot{e},n = 3) - (\dot{e}/e)\mathbf{I_{0-}}(e,\dot{e},n = 3).$$

From here, we are in a position to find a resolution for the wanted integral $\mathbf{I_0}(e,\dot{e},n = 3)$. After multiplying by $e/(e - \dot{e}) \neq 0$, we have:

(54) $$\mathbf{I_0}(e,\dot{e},n = 3) = [e/(e - \dot{e})]\mathbf{A_3}(e,\dot{e}) - [\dot{e}/(e - \dot{e})]\mathbf{I_{0-}}(e,\dot{e},n = 3).$$

To finish the solution process, we must replace the analytical representation (49) of the integral $\mathbf{A_3}(e,\dot{e})$ through the integral $\mathbf{I_{0-}}(e,\dot{e},n = 3)$:

(55) $$\mathbf{I_0}(e,\dot{e},n = 3) = (4\pi/5)e(e - \dot{e})^{-1}[1 - (e - \dot{e})^2]^{-3/2} + \{(3/5)e[1 - (e - \dot{e})^2](e - \dot{e})^{-1} - \dot{e}/(e - \dot{e})\}\, \mathbf{I_{0-}}(e,\dot{e},n = 3).$$

Further we compute the integral $\mathbf{I_{0+}}(e,\dot{e},n = 3)$ (see definition (35)):

(56) $\mathbf{I_{0+}}(e,\dot{e},n = 3) = \int_0^{2\pi}\{[1 + (e - \dot{e})cos\varphi] + \dot{e}cos\varphi\}[1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi =$

$= \mathbf{I_{0-}}(e,\dot{e},n = 3) + (\dot{e}/e)\mathbf{I_{0+}}(e,\dot{e},n = 3) - (\dot{e}/e)\mathbf{A_5}(e,\dot{e}),$

where we have taken into account the definitions (34), (35) and ((19), paper [5]) for the integrals $\mathbf{I_{0-}}(e,\dot{e},n = 3)$, $\mathbf{I_{0+}}(e,\dot{e},n = 3)$ and $\mathbf{A_5}(e,\dot{e})$, respectively. From the above derived relation (56) it immediately follows (after a multiplication by $e/\dot{e}$) an expression for the integral $\mathbf{A_5}(e,\dot{e})$ through the integrals $\mathbf{I_{0-}}(e,\dot{e},n = 3)$ and $\mathbf{I_{0+}}(e,\dot{e},n = 3)$:

(57) $\mathbf{A_5}(e,\dot{e}) \equiv \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi = (e/\dot{e})\mathbf{I_{0-}}(e,\dot{e},n = 3) - [(e - \dot{e})/\dot{e}]\mathbf{I_{0+}}(e,\dot{e},n = 3).$

Let us perform certain transformations of the assumed by us as "basic" integral $\mathbf{I_{0-}}(e,\dot{e},n = 3)$, in order to link it to some other integrals. And thus to establish the seeked representation of $\mathbf{I_{0+}}(e,\dot{e},n = 3)$ by means of $\mathbf{I_{0-}}(e,\dot{e},n = 3)$. Of course, we may substitute into (57) the already known solution (19) from paper [5] for $\mathbf{A_5}(e,\dot{e})$, and express in such a way $\mathbf{I_{0+}}(e,\dot{e},n = 3)$ solely by means of $\mathbf{I_{0-}}(e,\dot{e},n = 3)$. But we shall do this in a different manner.

(58) $\mathbf{I_{0-}}(e,\dot{e},n = 3) = \int_0^{2\pi}\{[1 + (e - \dot{e})cos\varphi] - (e - \dot{e})cos\varphi\}\, [1 + (e - \dot{e})cos\varphi]^{-4}\, d\varphi = \mathbf{A_3}(e,\dot{e}) +$

$+ 4(e - \dot{e})^2\int_0^{2\pi}(1 - cos^2\varphi)[1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi = \mathbf{A_3}(e,\dot{e}) + 4(e - \dot{e})^2\mathbf{A_5}(e,\dot{e}) + 4\int_0^{2\pi}[1 - (e - \dot{e})cos\varphi]\times$

$\times[1 + (e - \dot{e})cos\varphi][1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi - 4\int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi = \mathbf{A_3}(e,\dot{e}) -$

$- 4[1 - (e - \dot{e})^2]\mathbf{A_5}(e,\dot{e}) + 4\int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-4}\, d\varphi - 4\int_0^{2\pi}\{[1 + (e - \dot{e})cos\varphi] - 1\}\times$

$\times[1 + (e - \dot{e})cos\varphi]^{-4}\, d\varphi = - 3\mathbf{A_3}(e,\dot{e}) - 4[1 - (e - \dot{e})^2]\mathbf{A_5}(e,\dot{e}) + 8\mathbf{I_{0-}}(e,\dot{e},n = 3),$

where, evidently, we have used the definitions (16) and (19) from paper [5] for $\mathbf{A_3}(e,\dot{e})$ and $\mathbf{A_5}(e,\dot{e})$, and (34) for $\mathbf{I_{0-}}(e,\dot{e},n = 3) \equiv \mathbf{A_4}(e,\dot{e})$. Consequently, we have:

(59) $7\mathbf{I_{0-}}(e,\dot{e},n = 3) = 3\mathbf{A_3}(e,\dot{e}) + 4[1 - (e - \dot{e})^2]\mathbf{A_5}(e,\dot{e}).$

At present, it remains to substitute into this equality (59) the earlier derived results (49) for $\mathbf{A_3}(e,\dot{e})$ and (57) for $\mathbf{A_5}(e,\dot{e})$.The division by $4[1 - - (e - \dot{e})^2] \neq 0$ gives:

(60) $(7/4)[1 - (e - \dot{e})^2]^{-1}\mathbf{I_{0-}}(e,\dot{e},n = 3) - (3\pi/5)[1 - (e - \dot{e})^2]^{-5/2} - (9/20)\, \mathbf{I_{0-}}(e,\dot{e},n = 3) -$

$- (e/\dot{e})\mathbf{I_{0-}}(e,\dot{e},n = 3) = - [(e - \dot{e})/\dot{e}]\mathbf{I_{0+}}(e,\dot{e},n = 3).$

From here, it is easy to establish the following linear relation between the integrals $\mathbf{I_{0-}}(e,\dot{e},n = 3)$ and $\mathbf{I_{0+}}(e,\dot{e},n = 3)$:

(61) $\mathbf{I_{0+}}(e,\dot{e},n = 3) = (3\pi/5)\dot{e}(e - \dot{e})^{-1}[1 - (e - \dot{e})^2]^{-5/2} +$

$+ [\dot{e}/(e - \dot{e})]\{(9/20) + (e/\dot{e}) - (7/4)[1 - (e - \dot{e})^2]^{-1}\}\mathbf{I_{0-}}(e,\dot{e},n = 3).$

The derived above linear dependence (61) between the integrals $\mathbf{I_{0-}}(e,\dot{e},n = 3)$ and $\mathbf{I_{0+}}(e,\dot{e},n = 3)$ is remarkable with the conclusion that at least under the conditions (which are supposed during the evaluation (61)) $n = 3$, $e(u) \neq 0$, $\dot{e}(u) \neq 0$ and $e(u) - \dot{e}(u) \neq 0$ (i.e., Case 2.1.5.1) we already know the answer of the problem, which we are seeking for. Yes, the integrals $\mathbf{I_{0-}}(e,\dot{e},n = 3)$ and $I_{0+}(e,\dot{e},n = 3)$ are linearly depended! Such a finding is not surprising in view of the established earlier [8] analytical solutions for the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$ for integer powers $\boldsymbol{n}$ ($n = -1, 0, 1, 2, 3$). If we take the results for $n = 3$:

(62) $\mathbf{I_{0+}}(e,\dot{e},n = 3) = (\pi/4)(8e + 4e^3 - 12e^5 - 8\dot{e} - 32e^2\dot{e} + 45e^4\dot{e} + 52e\dot{e}^2 - 60e^3\dot{e}^2 - 24\dot{e}^3 + 30e^2\dot{e}^3 - 3\dot{e}^5)(e - \dot{e})^{-1}[1 - (e - \dot{e})^2]^{-9/2}$, (formula (6g) from paper [8]),

(63) $\mathbf{I_{0-}}(e,\dot{e},n = 3) = \pi[2 + 3(e - \dot{e})^2][1 - (e - \dot{e})^2]^{-7/2}$, (formula (6h) from paper [8]).

The linear relation (61) between the integrals $\mathbf{I_{0-}}(e,\dot{e},n = 3)$ and $\mathbf{I_{0+}}(e,\dot{e},n = 3)$ is fully consistent with the analytical expressions (62) and (63) for these functions of $e(u)$, $\dot{e}(u)$ and the (fixed) power $n = 3$. Similar conclusions about the existence of a linear dependence between $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$ can be made also for the other (fixed) integer values of the power n in the viscosity law $\eta = \beta\Sigma^n$: for $n = -1$ (see formulas (2g) and (2h) from paper [8]; for $n = 0$ (see formulas (3g) and (3h) from paper [8]); for $n = +1$ (see formulas (4g) and (4h) from paper [8]) and $n = +2$ (see formulas (5g) and (5h) from paper [8]). *We shall not enter here into a discussion about the explicit analytical form of the later pointed out linear functional dependences. Nor yet about their validity, as regards to the possible troubles for "peculiar" (i.e., vanishing some denominators of the expressions) values* $e(u) = 0$, $\dot{e}(u) = 0$, *and* $e(u) - \dot{e}(u) = 0$. We postpone such a debate for later considerations. Our dominant aim now is to prepare to solve the problem of the existence of linear relation between the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$ for *arbitrary* (physically reasonable) values of the power n. Of course, we remind that for every concrete accretion disc model, n remains a *preliminary fixed quantity* throughout the *whole* disc [1].

Let us compute the right-hand-side of the relation (12), in order to check its nullification for the *particular value* $n = 3$. We shall apply the results (55) and (61) for the integrals $\mathbf{I_0}(e,\dot{e},n = 3)$ and $\mathbf{I_{0+}}(e,\dot{e},n = 3)$, respectively:

(64) $[1 + 4(1 - e^2)(e - \dot{e})\dot{e}^{-1}]\mathbf{I_{0-}}(e,\dot{e},n = 3) - 4[(1 - e^2)(e - \dot{e})^2e^{-1}\dot{e}^{-1} + (2e - \dot{e})e^{-1}]\mathbf{I_{0+}}(e,\dot{e},n = 3) + 3\mathbf{I_0}(e,\dot{e},n = 3) = -(12\pi/5)(e^2 - 2e\dot{e} + \dot{e}^2 - e^4 + 2e^3\dot{e} - e^2\dot{e}^2 + 2e\dot{e} - \dot{e}^2)e^{-1}(e - \dot{e})^{-1}\times$

$$\times[1-(e-\dot{e})^2]^{-5/2} + (12\pi/5)e(1-e^2+2e\dot{e}-\dot{e}^2)(e-\dot{e})^{-1}[1-(e-\dot{e})^2]^{-5/2} +$$
$$+\boldsymbol{\{}(\dot{e}+4e-4\dot{e}-4e^3+4e^2\dot{e})\dot{e}^{-1}-4(e^2-e^4+2e^3\dot{e}-e^2\dot{e}^2)e^{-1}(e-\dot{e})^{-1}\times$$
$$\times\{(9/20)+(e/\dot{e})-(7/4)[1-(e-\dot{e})^2]^{-1}\}+(9/5)e[1-(e-\dot{e})^2](e-\dot{e})^{-1}-3\dot{e}(e-\dot{e})^{-1}\boldsymbol{\}}\mathbf{I_{0-}}(e,\dot{e},n=3)=$$
$$=\{20\dot{e}^2(e-\dot{e})[1-(e-\dot{e})^2]\}^{-1}\boldsymbol{\{}20\dot{e}(e-\dot{e})[1-(e-\dot{e})^2](4e-4e^3-3\dot{e}+4e\dot{e})+$$
$$+4\dot{e}(-e+e^3-2e^2\dot{e}+e\dot{e}^2)\{9\dot{e}[1-(e-\dot{e})^2]+20e[1-(e-\dot{e})^2]-35\dot{e}\}+36e\dot{e}^2(1-e^2+2e\dot{e}-\dot{e}^2)^2-$$
$$-60\dot{e}^3[1-(e-\dot{e})^2]\boldsymbol{\}}\mathbf{I_{0-}}(e,\dot{e},n=3)=\{20\dot{e}^2(e-\dot{e})[1-(e-\dot{e})^2]\}^{-1}\times 0\times\mathbf{I_{0-}}(e,\dot{e},n=3)\equiv 0.$$

To arrive to this zero result, we have taken into account (after some elementary algebra), that the multiplier of the integral $\mathbf{I_{0-}}(e,\dot{e},n = 3)$ into the square brackets is identically equal to zero. The same conclusion can be made also for the “free” term (i.e., the term without the integral $\mathbf{I_{0-}}(e,\dot{e},n = =3)$) in view of the identity: $(-e + e^3 - 2e^2\dot{e} + e\dot{e}^2 + e - e^3 + 2e^2\dot{e} - e\dot{e}^2) \equiv 0$.

That is why, the combination of these two equal to zero multipliers leads to the final nullification of the right-hand-side of the equality (12) for $n = 3$:

(65) $$\lim_{n\to 3}\{[(2n-5)+(n+1)(1-e^2)(e-\dot{e})\dot{e}^{-1}]\mathbf{I_{0-}}(e,\dot{e},n)-(n+1)[(1-e^2)(e-\dot{e})^2(e\dot{e})^{-1}+$$
$$+(2e-\dot{e})/e]\mathbf{I_{0+}}(e,\dot{e},n)+3\mathbf{I_0}(e,\dot{e},n)\}=0.$$

This evaluation (65) indicates that we are able to attempt to compute the integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n = 3)$ not only through a *direct* substitution $n = 3$ into its definition (4) (see also formula (37)), but also from the relation (12), using the limit transition $n \to 3$, in order to overcome the indeterminacy of the type 0/0. The reasoning to perform such a *duplicating* evaluation of the integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n = 3)$ is to show the *universality* of (12), i.e., that it remains valuable even in the case $n = 3$, despite of the necessity to interpret it through the limit transition $n \to 3$. With the invitation of the L’Hospital’s rule for resolving of the indeterminacies of the type 0/0. The conditions for applicability of this theorem are formulated in the textbooks on analysis and are also adduced for clearness in paper [5]. One of them (in our concrete task) is fulfilled by virtue of the established result (65). Other condition concerns the multiplier $(n - 3)(1 - e^2)$ into the left-hand-side of the equality (12), which, in fact, must be considered as a factor into the denominator in the right-hand-side of (12). If we regard (12) as a solution for the integral $\mathbf{I_{0,-4,+1}}(e,\dot{e},n = 3)$. Specifically:

(66) $$\lim_{n\to 3}\{\partial[(n-3)(1-e^2)]/\partial n\} = \lim_{n\to 3}(1-e^2) = 1-e^2 \neq 0,$$

because $|e(u)| < 1$. It is easily verified that the remaining conditions, for the applicability of the L’Hospital’s theorem, are available. But only if, at first, we have already computed the limit transition $n \to 3$ of the derivative with respect to n of the right-hand-side of the solution (12) for $\mathbf{I_{0,-4,+1}}(e,\dot{e},n)$. As we shall see now, the later evaluations *are not* too problemless for resolving. We just now start to resolve this task. We begin with the finding of the derivative with respect to the power n of the right-hand-side of the equality (12) and then take the limit $n \to 3$:

(67) $$\partial/\partial n\{[(2n-5) + (n+1)(1-e^2)(e-\dot{e})\dot{e}^{-1}]\mathbf{I_{0-}}(e,\dot{e},n) - (n+1)[(1-e^2)(e-\dot{e})^2(e\dot{e})^{-1} +$$
$$+ (2e-\dot{e})/e]\mathbf{I_{0+}}(e,\dot{e},n) + 3\mathbf{I_0}(e,\dot{e},n)\} \xrightarrow[n \to 3]{} [2 + (1-e^2)(e-\dot{e})\dot{e}^{-1}]\int_0^{2\pi}[1 + (e-\dot{e})cos\varphi]^{-4}\, d\varphi\; -$$
$$- [(1-e^2)(e-\dot{e})^2(e\dot{e})^{-1} + (2e-\dot{e})/e]\int_0^{2\pi}(1 + ecos\varphi)[1 + (e-\dot{e})cos\varphi]^{-5}\, d\varphi\; +$$
$$+ [1 + (1-e^2)(e-\dot{e})\dot{e}^{-1}]\int_0^{2\pi}[1 + (e-\dot{e})cos\varphi]^{-4}\mathbf{\{}ln\{(1 + ecos\varphi)[1 + (e-\dot{e})cos\varphi]^{-1}\}\mathbf{\}}\, d\varphi\; -$$
$$- 4[(1-e^2)(e-\dot{e})^2(e\dot{e})^{-1} + (2e-\dot{e})/e]\int_0^{2\pi}(1 + ecos\varphi)[1 + (e-\dot{e})cos\varphi]^{-5}\times$$
$$\times\mathbf{\{}ln\{(1 + ecos\varphi)[1 + (e-\dot{e})cos\varphi]^{-1}\}\mathbf{\}}\, d\varphi + 3\int_0^{2\pi}(1 + ecos\varphi)[1 + (e-\dot{e})cos\varphi]^{-4}\times$$
$$\times\mathbf{\{}ln\{(1 + ecos\varphi)[1 + (e-\dot{e})cos\varphi]^{-1}\}\mathbf{\}}\, d\varphi \equiv \mathbf{C}(e,\dot{e}).$$

In deriving of the above equality, we have taken into account the definitions (1), (2) and (3) for the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$, $\mathbf{I_{0+}}(e,\dot{e},n)$ and $\mathbf{I_0}(e,\dot{e},n)$, respectively. We also have used from the analysis the well-known differentiation formula:

(68) $$d(a^x)/dx = a^x ln(a); \qquad a > 0,$$

where the basis $a > 0$ does not depend on the variable x. From this rule immediately follows that if we have the constants $a > 0$, $b > 0$, y and z, then we can write:

(69) $$d/dx(a^{y+x}/b^{z+x}) \equiv d/dx(a^{y+x}b^{-z-x}) = (a^{y+x}/b^{z+x})ln(a) + (a^{y+x}/b^{z+x})[ln(b)]d(-x)/dx =$$
$$= (a^{y+x}/b^{z+x})[ln(a) - ln(b)] \equiv (a^{y+x}/b^{z+x})ln(a/b).$$

The above rule is applied, when the differentiation with respect to the power n of the integrands of the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$, $\mathbf{I_{0+}}(e,\dot{e},n)$ and $\mathbf{I_0}(e,\dot{e},n)$ has been performed. To continue the analytical evaluation of the right-hand-side of the relation (67) (which, after the transition $n \to 3$, we denote briefly by $\mathbf{C}(e,\dot{e})$), we have to return to certain auxiliary results, *derived especially for the present investigation*. They are published in papers [6] and [7], and are dealing with the analytical computations of the integrals $\mathbf{L_i}(e,\dot{e})$ ($\mathbf{i}$ = 0, 1, 2, 3) and $\mathbf{K_i}(e,\dot{e})$ ($\mathbf{i}$ = 1, 2, 3, 4, 5), (see their definitions (17) and (18), respectively). We do not rewrite here these solutions, and also the expressions for some of the *particular* values for the first and second arguments of $\mathbf{K_i}(e,\dot{e})$ ($\mathbf{i}$ = 1, 2, 3, 4, 5). Namely, $\mathbf{K_i}(e,0)$ ($\mathbf{i}$ = = 1, 2, 3, 4, 5) and $\mathbf{K_i}(e-\dot{e},0)$ ($\mathbf{i}$ = 3, 4, 5) [7]. We only refer to these (to some extend) long formulas in paper [7], in order to avoid the unnecessary overload of the our exposition. With the above remarks, we write from (67) that:

(70)
$$
\begin{aligned}
&\mathbf{C}(e,\dot{e}) = (e - e^3 + \dot{e} + e^2\dot{e})\dot{e}^{-1}\mathbf{A}_4(e,\dot{e}) + (-e + e^3 - 2e^2\dot{e} + e\dot{e}^2)\dot{e}^{-1}\mathbf{A}_5(e,\dot{e}) +\\
&+ \{e/[\dot{e}(e-\dot{e})]\}(-e + e^3 - 2e^2\dot{e} + e\dot{e}^2)\mathbf{A}_4(e,\dot{e}) - \{e/[\dot{e}(e-\dot{e})]\}(-e + e^3 - 2e^2\dot{e} + e\dot{e}^2)\mathbf{A}_5(e,\dot{e}) +\\
&+ (4e - 4e^3 - 3\dot{e} + 4e^2\dot{e})\dot{e}^{-1}\mathbf{K}_4(e,\dot{e}) - (4e - 4e^3 - 3\dot{e} + 4e^2\dot{e})\dot{e}^{-1}\mathbf{K}_4(e,\dot{e}) +\\
&+ (-4e + 4e^3 - 8e^2\dot{e} + 4e\dot{e}^2)\dot{e}^{-1}\mathbf{K}_5(e,\dot{e}) +\\
&+ \{e/[\dot{e}(e-\dot{e})]\}(-4e + 4e^3 - 8e^2\dot{e} + 4e\dot{e}^2)\mathbf{K}_4(e,\dot{e}) - \{e/[\dot{e}(e-\dot{e})]\}(-4e + 4e^3 - 8e^2\dot{e} +\\
&+4e\dot{e}^2)\mathbf{K}_5(e,\dot{e}) -\\
&- (-4e + 4e^3 - 8e^2\dot{e} + 4e\dot{e}^2)\dot{e}^{-1}\mathbf{K}_5(e-\dot{e},0) -\\
&- \{e/[\dot{e}(e-\dot{e})]\}(-4e + 4e^3 - 8e^2\dot{e} + 4e\dot{e}^2)\mathbf{K}_4(e-\dot{e},0) +\\
&+ \{e/[\dot{e}(e-\dot{e})]\}(-4e + 4e^3 - 8e^2\dot{e} + 4e\dot{e}^2)\mathbf{K}_5(e-\dot{e},0) + 3\mathbf{K}_4(e,\dot{e}) + 3[e/(e-\dot{e})]\mathbf{K}_3(e,\dot{e}) -\\
&- 3[e/(e-\dot{e})]\mathbf{K}_4(e,\dot{e}) - 3\mathbf{K}_4(e-\dot{e},0) - 3[e/(e-\dot{e})]\mathbf{K}_3(e-\dot{e},0) + 3[e/(e-\dot{e})]\mathbf{K}_4(e-\dot{e},0) =\\
&= [e/(e-\dot{e})][1-(e-\dot{e})^2]\mathbf{A}_5(e,\dot{e}) - [\dot{e}/(e-\dot{e})]\mathbf{A}_4(e,\dot{e}) + 4[e/(e-\dot{e})][1-(e-\dot{e})^2]\mathbf{K}_5(e,\dot{e}) -\\
&- 7[e/(e-\dot{e})]\mathbf{K}_4(e,\dot{e}) + 3[e/(e-\dot{e})]\mathbf{K}_3(e,\dot{e}) - 4[e/(e-\dot{e})][1-(e-\dot{e})^2]\mathbf{K}_5(e-\dot{e},0) +\\
&+ 7[e/(e-\dot{e})]\mathbf{K}_4(e-\dot{e},0) - 3[e/(e-\dot{e})]\mathbf{K}_3(e-\dot{e},0).
\end{aligned}
$$

At present, we are in a position to substitute into the last equality of the above relation the corresponding analytical evaluations for the integrals $\mathbf{A}_4(e,\dot{e})$ (formula (9) from paper [5]), $\mathbf{A}_5(e,\dot{e})$ (formula (19) from paper [5]), $\mathbf{K}_3(e,\dot{e})$ (formula (18) from paper [7]), $\mathbf{K}_4(e,\dot{e})$ (formula (20) from paper [7]), $\mathbf{K}_5(e,\dot{e})$ (formula (22) from paper [7]), $\mathbf{K}_3(e-\dot{e},0)$ (formula (31) from paper [7] with the replacement $e \to e - \dot{e}$), $\mathbf{K}_4(e-\dot{e},0)$ (formula (34) from paper [7] with the replacement $e \to e - \dot{e}$) and $\mathbf{K}_5(e-\dot{e},0)$ (formula (36) from paper [7] with the replacement $e \to e - \dot{e}$). Therefore:

(71)
$$
\begin{aligned}
&\mathbf{C}(e,\dot{e}) = (e-\dot{e})^{-1}\boldsymbol{\{}(\pi/4)e[1-(e-\dot{e})^2][8 + 24(e-\dot{e})^2 + 3(e-\dot{e})^4][1-(e-\dot{e})^2]^{-9/2} -\\
&- \pi\dot{e}[2 + 3(e-\dot{e})^2][1-(e-\dot{e})^2]^{-7/2} + 4e(\pi/4)[1-(e-\dot{e})^2](8 + 24e^2 + 3e^4 - 48e\dot{e} -\\
&- 12e^3\dot{e} + 24\dot{e}^2 + 18e^2\dot{e}^2 - 12e\dot{e}^3 + 3\dot{e}^4)[1-(e-\dot{e})^2]^{-9/2}ln\mathbf{Z}(e-\dot{e},0) + (4\pi/6)e[1-(e-\dot{e})^2]\times\\
&\times(-3e^4 + 9e^6 - 9e^8 + 3e^{10} - 4e^3\dot{e} - 10e^5\dot{e} + 32e^7\dot{e} - 18e^9\dot{e} - 6e^2\dot{e}^2 - 4e^4\dot{e}^2 - 35e^6\dot{e}^2 + 45e^8\dot{e}^2 -\\
&- 12e\dot{e}^3 - 16e^3\dot{e}^3 - 60e^7\dot{e}^3 + 39e^2\dot{e}^4 + 25e^4\dot{e}^4 + 45e^6\dot{e}^4 - 18e\dot{e}^5 - 16e^3\dot{e}^5 - 18e^5\dot{e}^5 + 3e^2\dot{e}^6 + 3e^4\dot{e}^6)\times\\
&\times\dot{e}^{-4}[1-(e-\dot{e})^2]^{-7/2} + (4\pi/12)e[1-(e-\dot{e})^2](6e^4 - 24e^6 + 36e^8 - 24e^{10} + 6e^{12} + 8e^3\dot{e} + 18e^5\dot{e} -\\
&- 102e^7\dot{e} + 118e^9\dot{e} - 42e^{11}\dot{e} + 12e^2\dot{e}^2 + e^4\dot{e}^2 + 88e^6\dot{e}^2 - 227e^8\dot{e}^2 + 126e^{10}\dot{e}^2 + 24e\dot{e}^3 + 16e^3\dot{e}^3 -\\
&- 35e^5\dot{e}^3 + 205e^7\dot{e}^3 - 210e^9\dot{e}^3 - 50\dot{e}^4 - 138e^2\dot{e}^4 + 48e^4\dot{e}^4 - 70e^6\dot{e}^4 + 210e^8\dot{e}^4 + 182e\dot{e}^5 - 40e^3\dot{e}^5 -\\
&- 16e^5\dot{e}^5 - 126e^7\dot{e}^5 - 55\dot{e}^6 - 4e^2\dot{e}^6 + 17e^4\dot{e}^6 + 42e^6\dot{e}^6 + 9e\dot{e}^7 - 3e^3\dot{e}^7 - 6e^5\dot{e}^7)\dot{e}^{-4}[1-(e-\dot{e})^2]^{-4}\times\\
&\times(1-e^2)^{-1/2} + (20\pi/12)e[1-(e-\dot{e})^2](10 + 11e^2 - 22e\dot{e} + 11\dot{e}^2)[1-(e-\dot{e})^2]^{-4} -\\
&- 7\pi e(2 + 3e^2 - 6e\dot{e} + 3\dot{e}^2)[1-(e-\dot{e})^2]^{-7/2}ln\mathbf{Z}(e-\dot{e},0) - (7\pi/3)e(-2e^3 + 4e^5 - 2e^7 - 3e^2\dot{e} - 5e^4\dot{e} +\\
&+ 8e^6\dot{e} - 6e\dot{e}^2 - 5e^3\dot{e}^2 - 12e^5\dot{e}^2 - 2\dot{e}^3 + 9e^2\dot{e}^3 + 8e^4\dot{e}^3 - 3e\dot{e}^4 - 2e^3\dot{e}^4)\dot{e}^{-3}[1-(e-\dot{e})^2]^{-5/2} -\\
&- (7\pi/3)e(2e^3 - 6e^5 + 6e^7 - 2e^9 + 3e^2\dot{e} + 4e^4\dot{e} - 17e^6\dot{e} + 10e^8\dot{e} + 6e\dot{e}^2 + e^3\dot{e}^2 + 13e^5\dot{e}^2 - 20e^7\dot{e}^2 - 11\dot{e}^3 -\\
&- 12e^2\dot{e}^3 + 3e^4\dot{e}^3 + 20e^6\dot{e}^3 + 17e\dot{e}^4 - 7e^3\dot{e}^4 - 10e^5\dot{e}^4 - 4\dot{e}^5 + 2e^2\dot{e}^5 + 2e^4\dot{e}^5)\dot{e}^{-3}[1-(e-\dot{e})^2]^{-3}\times\\
&\times(1-e^2)^{-1/2} - (14\pi/3)e[1-(e-\dot{e})^2]^{-5/2} - (7\pi/3)e(11 + 4e^2 - 8e\dot{e} + 4\dot{e}^2)[1-(e-\dot{e})^2]^{-3} +\\
&+ 3\pi e(2 + e^2 - 2e\dot{e} + \dot{e}^2)[1-(e-\dot{e})^2]^{-5/2}ln\mathbf{Z}(e-\dot{e},0) + 3\pi e(-e^2 + 2e^4 - e^6 - 2e\dot{e} - 2e^3\dot{e} + 4e^5\dot{e} -\\
&- 2e^2\dot{e}^2 - 6e^4\dot{e}^2 + 2e\dot{e}^3 + 4e^3\dot{e}^3 - e^2\dot{e}^4)\dot{e}^{-2}[1-(e-\dot{e})^2]^{-5/2} +\\
&+ 3\pi e(e^2 - 2e^4 + e^6 + 2e\dot{e} + e^3\dot{e} - 3e^5\dot{e} - 3\dot{e}^2 + 3e^4\dot{e}^2 + e\dot{e}^3 - e^3\dot{e}^3)\dot{e}^{-2}[1-(e-\dot{e})^2]^{-2}(1-e^2)^{-1/2} +\\
&+ 9\pi e[1-(e-\dot{e})^2]^{-2} - (4\pi/48)e[1-(e-\dot{e})^2][(96 + 288(e-\dot{e})^2 + 36(e-\dot{e})^4][1-(e-\dot{e})^2]^{-9/2}\times\\
&\times ln\mathbf{Z}(e-\dot{e},0) - (4\pi/48)e[1-(e-\dot{e})^2]\{-200 - 312(e-\dot{e})^2 - 21(e-\dot{e})^4 + [200 + 220(e-\dot{e})^2]\times\\
&\times[1-(e-\dot{e})^2]^{1/2}\}[1-(e-\dot{e})^2]^{-9/2} + (7\pi/6)e[12 + 18(e-\dot{e})^2][1-(e-\dot{e})^2]^{-7/2}ln\mathbf{Z}(e-\dot{e},0) +\\
&+ (7\pi/6)e\{-22 - 15(e-\dot{e})^2 + [22 + 8(e-\dot{e})^2][1-(e-\dot{e})^2]^{1/2}\}[1-(e-\dot{e})^2]^{-7/2} -\\
&- (3\pi/2)e[4 + 2(e-\dot{e})^2][1-(e-\dot{e})^2]^{-5/2}ln\mathbf{Z}(e-\dot{e},0) - (3\pi/2)e\{-6 - (e-\dot{e})^2 + 6[1-(e-\dot{e})^2]^{1/2}\}\times\\
&\times[1-(e-\dot{e})^2]^{-5/2}\boldsymbol{\}}.
\end{aligned}
$$

In the above equality we have used the notation $\mathbf{Z}(e,\dot{e})$, introduced in the paper [7] (formula (23) in paper [7]). If we accept for the first argument of this function the difference $e(u) - \dot{e}(u)$, and a constant zero value for the second argument, it is easy to see (formula (24) from paper [7] with the replacement $e \to e - \dot{e}$) that:

$$\mathbf{Z}(e-\dot{e},0) = 2\{2 - 3(e-\dot{e})^2 + (e-\dot{e})^4 - 2[1-(e-\dot{e})^2]^{3/2}\}(e-\dot{e})^{-2}\{1-[1-(e-\dot{e})^2]^{1/2}\}^{-1}. \tag{72}$$

The long expression (71) for the function $\mathbf{C}(e,\dot{e})$ may be simplified, if we notice that the coefficient, multiplying the sum of the six terms with the logarithmic function $ln\mathbf{Z}(e - \dot{e},0)$, *is exactly equal to zero*. Even more: the coupling of such terms by triplets with common coefficients shows that the later are also with zero values. Let us prove this statement. Combining in (71) the 3rd, the 7th and the 12th terms together, and also the 16th, the 18th the 20th terms by triplets, we obtain also a zero sum.

Rejecting the above mentioned six terms (because of their zero contribution), we further simplify the expression for $\mathbf{C}(e,\dot{e})$, which already does not contain any logarithmic functions. After some algebraic transformations of the remaining 15 terms, we arrive at the final conclusion:

$$\begin{aligned}&\lim_{n\to 3}\Big\{\partial/\partial n\{[(2n-5)+(n+1)(1-e^2)(e-\dot{e})\dot{e}^{-1}]\mathbf{I}_{\mathbf{0-}}(e,\dot{e},n)-(n+1)[(1-e^2)(e-\dot{e})^2(e\dot{e})^{-1}+\\&+(2e-\dot{e})/e]\mathbf{I}_{\mathbf{0+}}(e,\dot{e},n)+3\mathbf{I}_{\mathbf{0}}(e,\dot{e},n)\}\Big\}\equiv\mathbf{C}(e,\dot{e})=\pi(1-e^2)(e-\dot{e})(-2e^3+6e^5-6e^7+2e^9-2e^2\dot{e}-\\&-8e^4\dot{e}+22e^6\dot{e}-12e^8\dot{e}-2e\dot{e}^2-e^3\dot{e}^2-27e^5\dot{e}^2+30e^7\dot{e}^2-2\dot{e}^3-e^2\dot{e}^3+8e^4\dot{e}^3-40e^6\dot{e}^3+7e\dot{e}^4+8e^3\dot{e}^4+\\&+30e^5\dot{e}^4-3\dot{e}^5-6e^2\dot{e}^5-12e^4\dot{e}^5+e\dot{e}^6+2e^3\dot{e}^6)\dot{e}^{-4}[1-(e-\dot{e})^2]^{-7/2}+2\pi e^4(1-e^2)\dot{e}^{-4}(1-e^2)^{-1/2}.\end{aligned} \tag{73}$$

Taking into account the transition (66), we have from the relation (12) (after a division by $(1 - e^2) \neq 0$), that:

$$\begin{aligned}&\lim_{n\to 3}\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e},n)\equiv\lim_{n\to 3}\Big\{\int_0^{2\pi}(1+ecos\varphi)^{n-4}[1+(e-\dot{e})cos\varphi]^{-n-1}\,d\varphi\Big\}=\pi(e-\dot{e})(-2e^3+6e^5-6e^7+\\&+2e^9-2e^2\dot{e}-8e^4\dot{e}+22e^6\dot{e}-12e^8\dot{e}-2e\dot{e}^2-e^3\dot{e}^2-27e^5\dot{e}^2+30e^7\dot{e}^2-2\dot{e}^3-e^2\dot{e}^3+8e^4\dot{e}^3-40e^6\dot{e}^3+\\&+7e\dot{e}^4+8e^3\dot{e}^4+30e^5\dot{e}^4-3\dot{e}^5-6e^2\dot{e}^5-12e^4\dot{e}^5+e\dot{e}^6+2e^3\dot{e}^6)\dot{e}^{-4}[1-(e-\dot{e})^2]^{-7/2}+\\&+2\pi e^4\dot{e}^{-4}(1-e^2)^{-1/2}=\int_0^{2\pi}(1+ecos\varphi)^{-1}[1+(e-\dot{e})cos\varphi]^{-4}\,d\varphi\equiv\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e},n=3).\end{aligned} \tag{74}$$

The above expression coincides with the expression for the integral $\mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e},n = 3) \equiv \mathbf{J}_{\mathbf{4}}(e,\dot{e})$ (formula (37), which is, in fact, the result (47), derived in the paper [5]). This means that the transition $n \to 3$ in the solution (13) is continuous. And it is possible to use this linear relation even for $n = 3$, having in mind that we have to apply, in this connection, the L'Hospital's rule for resolving of indeterminacies of the type 0/0.

2.1.5.2. Case $n = 3$, $e(u) \neq 0$, $\dot{e}(u) = e(u) \neq 0$ => $e(u) - \dot{e}(u) = 0$

A direct computation from the definition (4) gives that:

(75) $$\mathbf{I}_{0,-4,+1}(e, \dot{e} = e, n = 3) \equiv \int_0^{2\pi} (1 + e\cos\varphi)^{-1}\, d\varphi = 2\pi(1 - e^2)^{-1/2} = \mathbf{A}_1(e,0),$$

(see, for example, formula (20) from paper [5]).

This result follows also from the above just derived expression (74), if we substitute into it $\dot{e}(u) = e(u)$. It is also in agreement with the relation (12), because (74) is derived as a consequence from (12) in the limit $n \to 3$.

2.1.6. Case $n = 3$, $e(u) \neq 0$, $\dot{e}(u) = 0$ => $e(u) - \dot{e}(u) \neq 0$

A direct computation from the definition (4) gives that:

(76) $$\mathbf{I}_{0,-4,+1}(e, \dot{e} = e, n = 3) = \mathbf{A}_5(e,0) = (\pi/4)(8 + 24e^2 + 3e^4)(1 - e^2)^{-9/2},$$

(formula (24) from paper [5]).

In deriving of the above expression (76), we have at first taken the limit $\dot{e}(u) \to 0$, and after then we have performed the transition $n \to 0$. The solutions (75) and (76) coincide with the solutions $\mathbf{A}_1(e,0)$ and $\mathbf{A}_5(e,0)$, respectively, and we note that the laters do not depend on the power n in the viscosity law $\eta = \beta\Sigma^n$. We now shall show that we may change the order of the transitions: at first we may take into (4) the transition $n \to 3$ and after that substitute $\dot{e}(u) = 0$. The final result will be the same as (76). In the expression (74) the transition $n \to 3$ is already performed and it remains to evaluate it in the limit $\dot{e}(u) \to 0$. Performing into the first term the multiplication by $(e - \dot{e}) \neq 0$ (in the our Case 2.1.6 $e(u) - \dot{e}(u) \neq 0$!) and reducing to a common denominator the two terms of the solution (74), we want, in fact, to evaluate the limit:

(78) $$\pi(1 - e^2)^{-1/2} \lim_{\dot{e}(u) \to 0} \{\{(-2e^4 + 6e^6 - 6e^8 + 2e^{10} - 14e^5\dot{e} + 28e^7\dot{e} - 14e^9\dot{e} + 7e^4\dot{e}^2 - 49e^6\dot{e}^2 + 42e^8\dot{e}^2 + 35e^5\dot{e}^3 - 70e^7\dot{e}^3 + 2\dot{e}^4 + 8e^2\dot{e}^4 + 70e^6\dot{e}^4 - 10e\dot{e}^5 - 14e^3\dot{e}^5 - 42e^5\dot{e}^5 + 3\dot{e}^6 + 7e^2\dot{e}^6 + 14e^4\dot{e}^6 - e\dot{e}^7 - 2e^3\dot{e}^7)(1 - e^2)^{1/2} + 2e^4[1 - (e - \dot{e})^2]^{7/2}\}\dot{e}^{-4}[1 - (e - \dot{e})^2]^{-7/2}\}.$$

To apply the L'Hospital's rule for evaluating of indeterminasies of the type 0/0, we must compute the derivatives with respect to $\dot{e}(u)$ from the denominator and the dominator of the above expression (78).

It is easily checked that the other conditions for the application of the L'Hospital's rule (see paper [5]) are also fulfilled. It turns out, that this rule has to be used *four* times, because only after the *fourth* differentiation with respect to $\dot{e}(u)$ of the denominator $\dot{e}^4[1 - (e - \dot{e})^2]^{7/2}$ into (78) ensures non-zero value for $\dot{e}(u) = 0$. We here *temporarily* neglect the multiplier $\pi(1 - e^2)^{-1/2}$ into the left-hand-side of the equality (78), because it does not make sense under the limit transition $\dot{e}(u) \to 0$. Therefore, we successively compute the following derivatives with respect to $\dot{e}(u)$ under the transition $\dot{e}(u) \to 0$:

(79) $$\lim_{\dot{e}(u) \to 0} \{\partial/\partial\dot{e}\{\dot{e}^4[1 - (e - \dot{e})^2]^{7/2}\}\} = 0.$$

(80) $$\lim_{\dot{e}(u) \to 0} \{\partial/\partial\dot{e}\{4\dot{e}^3[1 - (e - \dot{e})^2]^{7/2} + 7\dot{e}^4(e - \dot{e})[1 - (e - \dot{e})^2]^{5/2}\}\} = 0.$$

(81) $$\lim_{\dot{e}(u) \to 0} \{\partial/\partial\dot{e}\{12\dot{e}^2[1 - (e - \dot{e})^2]^{7/2} + 56\dot{e}^3(e - \dot{e})[1 - (e - \dot{e})^2]^{5/2} - 7\dot{e}^4[1 - (e - \dot{e})^2]^{5/2} + \\ + 35\dot{e}^4(e - \dot{e})^2[1 - (e - \dot{e})^2]^{3/2}\}\} = 0.$$

Finally, we compute analytically that:

(82) $$\lim_{\dot{e}(u) \to 0} \{\partial/\partial\dot{e}\{24\dot{e}[1 - (e - \dot{e})^2]^{7/2} + 252\dot{e}^2(e - \dot{e})[1 - (e - \dot{e})^2]^{5/2} - 84\dot{e}^3[1 - (e - \dot{e})^2]^{5/2} + \\ + 420\dot{e}^3(e - \dot{e})^2[1 - (e - \dot{e})^2]^{3/2} - 105\dot{e}^4(e - \dot{e})[1 - (e - \dot{e})^2]^{3/2} + 105\dot{e}^4(e - \dot{e})^3[1 - (e - \dot{e})^2]^{1/2}\}\} = \\ = 24(1 - e^2)^{7/2}.$$

Therefore, we have to differentiate with respect to $\dot{e}(u)$ *four times*, until we arrive at an expression in the *denominator*, which tends to non-zero value, when $\dot{e}(u) \to 0$.

Let us now compute the limits of the derivatives of the *nominator* of the expression (78), when $\dot{e}(u)$ approaches zero:

(83) $$\lim_{\dot{e}(u) \to 0} \{\partial/\partial\dot{e}\{(-2e^4 + 6e^6 - 6e^8 + 2e^{10} - 14e^5\dot{e} + 28e^7\dot{e} - 14e^9\dot{e} + 7e^4\dot{e}^2 - 49e^6\dot{e}^2 + 42e^8\dot{e}^2 + 35e^5\dot{e}^3 - \\ - 70e^7\dot{e}^3 + 2\dot{e}^4 + 8e^2\dot{e}^4 + 70e^6\dot{e}^4 - 10e\dot{e}^5 - 14e^3\dot{e}^5 - 42e^5\dot{e}^5 + 3\dot{e}^6 + 7e^2\dot{e}^6 + 14e^4\dot{e}^6 - e\dot{e}^7 - 2e^3\dot{e}^7)\times \\ \times(1 - e^2)^{1/2} + 2e^4[1 - (e - \dot{e})^2]^{7/2}\}\} = 0,$$

(84) $$\lim_{\dot{e}(u) \to 0} \{\partial/\partial\dot{e}\{(-14e^5 + 28e^7 - 14e^9 + 14e^4\dot{e} - 98e^6\dot{e} + 84e^8\dot{e} + 105e^5\dot{e}^2 - 210e^7\dot{e}^2 + \ldots)\times(1 - e^2)^{1/2} + \\ + (14e^5 - 28e^7 + 14e^9 - 14e^4\dot{e} + 84e^6\dot{e} - 70e^8\dot{e} - 84e^5\dot{e}^2 + 140e^7\dot{e}^2 + \ldots)[1 - (e - \dot{e})^2]^{1/2}\}\} = 0.$$

(85) $$\lim_{\dot{e}(u) \to 0} \{\partial/\partial\dot{e}\{(14e^4 - 98e^6 + 84e^8 + 210e^5\dot{e} - 420e^7\dot{e} + \ldots)(1 - e^2)^{1/2} + (14e^4 + 98e^6 - 84e^8 - \\ - 196e^5\dot{e} + \ldots)[1 - (e - \dot{e})^2]^{1/2}\}\} = 0,$$

(86) $$\lim_{\dot{e}(u) \to 0} \{\partial/\partial\dot{e}\{(210e^5 - 420e^7 + 48\dot{e} + 192e^2\dot{e} + 1680e^6\dot{e} - \ldots)(1 - e^2)^{1/2} + (-196e^5 + 336e^7 + 196e^4\dot{e} - \\ - 1008e^6\dot{e} + \ldots)[1 - (e - \dot{e})^2]^{1/2} + (-14e^5 + 98e^7 - 84e^9 + 14e^4\dot{e} - 294e^6\dot{e} + 420e^8\dot{e} + \ldots)\times \\ \times[1 - (e - \dot{e})^2]^{-1/2}\}\} = 6(8 + 24e^2 + 3e^4)(1 - e^2)^{-1/2}.$$

This is the *fourth* differentiation of the *nominator* of the expression (78). Consequently, the L'Hospital's rule for resolving of indeterminacies of the type 0/0, enables us to compute the two-limits transition:

(87) $$\lim_{\dot{e}(u) \to 0} [\lim_{n \to 3} \mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e},n)] = \lim_{\dot{e}(u) \to 0} \mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e},n = 3)] = \mathbf{I}_{\mathbf{0,-4,+1}}(e,\dot{e} = 0,n = 3)] = \\ = (6\pi/24)(8 + 24e^2 + 3e^4)(1 - e^2)^{-1}(1 - e^2)^{-7/2} = (\pi/4)(8 + 24e^2 + 3e^4)(1 - e^2)^{-9/2},$$

where the above solution (87) follows from the equalities (82) and (86). And also we have recovered the multiplier $\pi(1 - e^2)^{-1/2}$, according to the expression (78). The evaluation (87) coincides with the right-hand-side of the solution (76) and implies that *no matter* which limit transition $\dot{e}(u) \to 0$

or $n \to 3$ will be taken first. That is to say, the two-limits transitions $\lim_{\dot{e}(u) \to 0} \lim_{n \to 3}$ and $\lim_{n \to 3} \lim_{\dot{e}(u) \to 0}$ are equivalent in the considered by us situations. This also confirms again, that the relation (12) may be used for evaluation of the integral $\mathbf{I}_{0,-4,+1}(e,\dot{e},n)$ in the cases when $n = 3$ and/or $\dot{e}(u) =$ $= 0$ through the application of the L'Hospital's rule for computing of indeterminacies of the type 0/0.

2.1.7. Case $n = 3$, $e(u) = 0$, $\dot{e}(u) \neq 0$ => $e(u) - \dot{e}(u) \neq 0$

A direct computation from the definition (4) gives:

$$\text{(88)} \quad \mathbf{I}_{0,-4,+1}(e = 0,\dot{e},n = 3) \equiv \int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-4}\, d\varphi = \mathbf{A_4}(-\dot{e},0) = \pi(2 + 3\dot{e}^2)(1 - \dot{e}^2)^{-7/2},$$

(formula (23) from paper [5] with the replacement $e(u) \to -\dot{e}(u)$).

We may also evaluate this integral by another way, using the solution (74), where the transition $n \to 3$ is already performed. And where we are allowed directly to set $e(u) = 0$ (simultaneously preserving $\dot{e}(u) \neq 0$), because $e(u)$ does not take place as a factor into the denominators. The result is:

$$\text{(89)} \quad \lim_{e(u) \to 0} [\, \lim_{n \to 3} \mathbf{I}_{0,-4,+1}(e,\dot{e},n)] = -\pi\dot{e}(-2\dot{e}^3 - 3\dot{e}^5)\dot{e}^{-4}(1 - \dot{e}^2)^{-7/2} = \pi(2 + 3\dot{e}^2)(1 - \dot{e}^2)^{-7/2},$$

which coincides with the above evaluation (88). Consequently, we again arrive at the conclusion that *no matter* which of the transitions $n \to 3$ or $e(u) \to 0$ must be realized at first. The expression (12) also may be useful (i.e., to make sense) for the analytical evaluation of the integral $\mathbf{I}_{0,-4,+1}(e,\dot{e},n)$ for $n = 3$ and/or $e(u) = 0$, if the corresponding two-limits $n \to 3$ and $e(u) \to$ $\to 0$ are performed.

2.1.8. Case $n = 3$, $e(u) = \dot{e}(u) = 0$ => $e(u) - \dot{e}(u) = 0$

Obviously, in this most simple case $\mathbf{I}_{0,-4,+1}(e = 0,\dot{e} = 0,n = 3) = 2\pi$. The same result follows from the expressions (76) and (88), if we take $e(u) \to 0$, or $\dot{e}(u) \to 0$, respectively.

All the above considerations, made in the **Cases 2.1.1 − 2.1.8**, support the statement that the *linear relation* (12) may be used also in the situations when some or all of the quantities $e(u)$, $\dot{e}(u)$ and $n - 3$ are equal to zero. It is enough only to apply the L'Hospital's rule (theorem) for resolving of indeterminacies of the type 0/0. It seems out, that there is *no matter* what must be the order of performing of the needed transitions $e(u) \to 0$, $\dot{e}(u) \to 0$ and $n \to 3$.

3. Conclusions

In the present paper we have resolved analytically the integral $\mathbf{I}_{0,-4,+1}(e,\dot{e},n)$, given by the definition (4). A similar approach for an analytical evaluation of the other integral $\mathbf{I}_{0,-2,+3}(e,\dot{e},n)$, described by the definition (5), will be applied in a forthcoming paper [9]. Such calculations split into many particular cases. This situation is caused by the vanishing of the denominators of some terms in the final or/and intermediate results for certain values of the eccentricity $e(u)$, its derivative $\dot{e}(u)$ and the power n. It is remarkable that all these solutions can be expressed by means of a single common formula. The essential point is that such divergences may be overcome with the help of the L'Hospital's rule for resolving of indeterminacies of the type 0/0. For this reason, the application of the solutions into the subsequent calculations is simplified to some extent, because there is not already need to consider every case in a separate way. Of course, having in mind the corresponding limit transitions, when we have dealing with the singular points. The generalized in such a manner solution for the integral $\mathbf{I}_{0,-4,+1}(e,\dot{e},n)$ is given by the formula (12). The corresponding to the integral $\mathbf{I}_{0,-2,+3}(e,\dot{e},n)$ solution is derived in paper [9].

The basic motivation to establish the analytical solutions of the integrals $\mathbf{I}_{0,-4,+1}(e,\dot{e},n)$ (definition (4)) and $\mathbf{I}_{0,-2,+3}(e,\dot{e},n)$ (definition (5)) is to give the answer of the question whether the integrals $\mathbf{I}_{0-}(e,\dot{e},n)$ (definition (1)) and $\mathbf{I}_{0+}(e,\dot{e},n)$ (definition (2)) are linearly dependent functions of $e(u)$, $\dot{e}(u)$ and n or not. The standard approach to resolve this problem is to compute the corresponding Wronski determinant and to evaluate its equalization/non-equalization to zero value. In the process of realization of this procedure, there arises the necessity of knowledge of the analytical solutions of these integrals $\mathbf{I}_{0,-4,+1}(e,\dot{e},n)$ and $\mathbf{I}_{0,-2,+3}(e,\dot{e},n)$. It is worth to note, that in the present investigation we already encounter with the property that for *integer n* ($n = -1, 0, 1, 2, 3$) the integrals $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$ are linearly dependent functions. In particular, formula (61) clearly demonstrates such a linear relation for $n = 3$. Therefore, we have a hint to expect also the existence of linear dependencies between $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$ in the general case, including the *non-integer* values of the power n. Such an expectation follows from the property that the viscosity law $\eta = \beta\Sigma^{n}$ does not impose or require any physically motivated separations of the powers n (for different families of models of Lyubarskij et al. [1]) into *integer* and *non-integer* values. That is to say, between models with (fixed) *integer n* and models with (fixed) *non-integer n*.

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АНАЛИТИЧНО ПРЕСМЯТАНЕ НА ДВА ИНТЕГРАЛА, ВЪЗНИКВАЩИ В ТЕОРИЯТА НА ЕЛИПТИЧНИТЕ АКРЕЦИОННИ ДИСКОВЕ. IV. РЕШАВАНЕ НА ЕДИН ИНТЕГРАЛ, ОБЕЗПЕЧАВАЩ ОЦЕНЯВАНЕТО НА ПРОИЗВОДНИТЕ, ВЛИЗАЩИ В ДЕТЕРМИНАНТАТА НА ВРОНСКИ

Д. Димитров

Резюме

Настоящата статия се занимава с аналитичното пресмятане на определения интеграл

$\int_0^{2\pi}(1 + e\cos\varphi)^{n-4}[1 + (e - \dot{e})\cos\varphi]^{-n-1}\, d\varphi$, където $e(u)$ са

ексцентрицитетите на орбитите на частиците, $\dot{e}(u) \equiv de(u)/du$, $u \equiv \ln(p)$, като p е фокалният параметър на съответните елиптични орбити на частиците. Параметърът n е степента в закона за вискозитета $\eta = \beta\Sigma^{n}$, където Σ е повърхностната плътност на акреционния диск и φ е азимуталният ъгъл. Ние сме извършили изчисленията при следните три ограничения: (*i*) $|e(u)| < 1$, (*ii*) $|\dot{e}(u)| < 1$ и (*iii*) $|e(u) - \dot{e}(u)| < 1$. Те са физически мотивирани от възприетия за нашите разглеждания модел на *стационарни* елиптични акреционни дискове на Любарски и др. [1]. Голям брой частни случаи, възникващи поради сингулярното поведение на някои членове за дадени значения на $e(u)$, $\dot{e}(u)$, тяхната разлика $e(u) - \dot{e}(u)$ и степенния показател n, са детайлно изчислени. Тези пресмятания са извършени по два способа: (*i*) чрез *директно* полагане на сингулярното значение в първоначалната дефиниция на интиграла, и (*ii*) чрез граничен преход към това сингулярно значение във вече оценения аналитичен израз за интеграла, получен за регулярни стойности на съответните променливи. В последния случай е твърде полезно прилагането на правилото на Льопитал за решаването на неопределености от вида 0/0. Двата подхода дават едни и същи резултати във всеки проверяван случай, което осигурява щото преходът през сингулярното значение да е непрекъснат. Това означава, че аналитичните решения за всички (сингулярни и несингулярни) случаи могат да бъдат комбинирани в една единствена формула. Такова едно описание на решението на горенаписания интеграл, е твърде удобно за случая, когато тази формула се прилага за проверяването на линейната зависимост/независимост на коефициентите, влизащи в членовете на динамичното уравнение на елиптичния акреционен диск.