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ANALYTICAL COMPUTATION OF TWO INTEGRALS, APPEARING IN THE THEORY OF ELLIPTICAL ACCRETION DISCS. II. SOLVING OF SOME AUXILIARY INTEGRALS, CONTAINING LOGARITHMIC FUNCTIONS INTO THEIR INTEGRANDS

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Abstract

This paper is a part of the investigations, dealing with the mathematical structure of the stationary elliptical accretion discs in the model of Lyubarskij et al. [1], i.e., discs for which all the apse lines of the particle orbits are in line with each other. The main point of the adopted approach is to find linear relations between the integrals, entering into the dynamical equation for these objects. They will enable us to eliminate these complicate (and, generally speaking, unknown analytically) functions of the eccentricity $\boldsymbol{e(u)}$ *and its derivative* $\boldsymbol{\dot{e}(u) \equiv de(u)/du}$ *of the individual orbits. Here* $\boldsymbol{u \equiv ln(p)}$*, where* $\boldsymbol{p}$ *is the focal parameter of the corresponding accretion disc particle orbit. During the process of realization of this program, we strike with the necessity to find analytical evaluations for two kinds of integrals:*

$$L_i(e,\dot{e}) \equiv \int_0^{2\pi} [ln(1 + e\cos\varphi)](1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-i}\, d\varphi,\ (i = 0,\dots, 3),\ \text{and}\ K_j(e,\dot{e}) \equiv \int_0^{2\pi} [ln(1 + e\cos\varphi)]\times$$

$$\times[1 + (e - \dot{e})\cos\varphi]^{-j}\, d\varphi,\ (j = 1,\ \dots,\ 5).$$

In the present investigation we find recurrence relations, allowing us to compute the integrals $\boldsymbol{L_i(e,\dot{e})}$*, (*$\boldsymbol{i}$ *= 1, 2, 3), under the condition that the integrals* $\boldsymbol{L_{i-1}(e,\dot{e})}$ *and* $\boldsymbol{K_i(e,\dot{e})}$ *are already known. Conversely, computations of the integrals* $\boldsymbol{K_j(e,\dot{e})}$*, (j = 1, …, 5), through the recurrence dependences do not require the knowledge of the analytical solutions of the integrals* $\boldsymbol{L_i(e,\dot{e})}$*, (*$\boldsymbol{i}$ *= 0, …, 3). In view of the fact that the integrals* $\boldsymbol{L_0(e)}$ *(it does not depend on* $\boldsymbol{\dot{e}(u)}$*) and* $\boldsymbol{K_1(e,\dot{e})}$ *serve as "starting--points" into the corresponding recurrence relations, we have find analytical expressions for them. The solution of the full set of analytical evaluations of* $\boldsymbol{L_i(e,\dot{e})}$*, (*$\boldsymbol{i}$ *= 1, 2, 3), and* $\boldsymbol{K_j(e,\dot{e})}$*, (j = 2, …, 5), will be given elsewhere [7].*

1. Introduction

The present paper continues a series of investigations, devoted to the simplification of the dynamical equation of the elliptical accretion discs. Especially, the considerations are constrained to a specific model, developed by Lyubarskij at al. [1]. The essential property of this model is that the all elliptical particle orbits are sharing a common longitude of periastron. The other restriction, which we impose on the adopted elaboration, is that the accretion flow is *stationary*. That is why, the dynamical equation, with which we are dealing, governs the *stationary* space structure of the disc. We remind that the particle orbits at different parts of the disc, may have different eccentricities $e(u)$, respectively. Here with the variable u we denote the logarithm of the focal parameter p of the corresponding elliptical orbit: $u \equiv ln(p)$. Also we shall often use the notation $\dot{e}(u) \equiv de(u)/du$ for the first ordinary derivative of the eccentricity $e(u)$ with respect to u. The way we proceed, to reveal the mathematical structure of the above mentioned equation, is to eliminate certain definite integrals over the azimuthal angle φ. They are functions of $e(u)$, $\dot{e}(u)$ and the power n in the viscosity law $\eta = \beta\, \Sigma^n$ (β is a constant, η is the viscosity and Σ is the surface density of the accretion disc). The procedure of reducing of the number of these integrals, by means of establishing of linear relations between them, is described and applied in earlier papers ([2], [3] and the references therein). Until now, the question: if three of these functions $\mathbf{I_3}(e,\dot{e},n)$, $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$ are linearly independent **or** not, still remains open (for definitions of these three integrals see [2] and [3]). The standard method to check which of these two alternative cases is available, is to compute the corresponding Wronskian. The procedure includes evaluation of some derivatives with respect to $e(u)$ or $\dot{e}(u)$ of the above mentioned integrals. In turn, this leads to appearing of two new integrals, for which we also have to find analytical solutions. In the course of realizing of this computational scheme, we, at first, must have available analytical expressions of given auxiliary integrals. In the preceding paper [4], we have given the solutions of such integrals, when their integrands do not include logarithmic functions of $e(u)$ or $\dot{e}(u)$. The present investigation deals just with this complementary case. It will be seen from the following exposition, that such integrals arise, when we obtain formulas, containing into their denominators factors, vanishing for some integer values of the power n. But from a physical point of view, we do not expect that the integer numbers n have "special" meanings in the considered accretion disc theory. And it is reasonable to check the "problem" formulas for their behaviour, when n approaches the "singular" value. It turns out, that the corresponding nominators also tend to zero, "compensating" the divergent (at the first glance) character of the analytical expression. As usual, it is instructive to apply in this situation the L'Hospital's rule for resolving of indeterminacies of the type 0/0. In turn, similar computational scheme implies the necessity of finding the partial derivatives with respect to the power n. More specifically, for the considered by us integrals, we shall compute derivatives like:

(1) $\partial[(1 + e\cos\varphi)^n]/\partial n = n\, ln(1 + e\cos\varphi)$,

according to the well-known rules from the differential calculus. In the above formula we take into account that the eccentricity $e(u)$ (or the difference $e(u) - \dot{e}(u)$, which may stand in the place of $e(u)$) and the azimuthal angle φ do not depend on n. Of course, the considered model of elliptical accretion discs [1] keeps fixed the power n (i.e., the viscosity law $\eta = \beta\, \Sigma^{\,n}$ remains valid throughout the entire disc) for every concrete accretion disc. The mathematical variability of n in the equality (1) should be supposed physically as a change/transition from one accretion flow (with a given fixed power n) to another accretion flow (with other, also fixed, but a little different value of n).

2. Integrands, including as a factor logarithmic function

The available handbooks, which we had considered, in order to find already computed analytical expressions for the integrals, representing an interest for us, do not give a direct answer to the task. We do not strike *only* with the incompleteness of the lists of the cited formulas, but also with the need to obtain evaluations of the integrals, which are valid for special choices (i.e., restrictions on the domains) of the parameter space, characterizing them. Probably, the specialization of the considered by us problem, leads to two possible situations:

(*i*) The integrals, for which we are seeking, are too "specialized", in view of the circumstance that the considered problem also treats too "narrow" aspects of the physical/mathematical theory. Correspondingly, such solutions of the integrals remain, as a rule, out of the attention of the compilers of the reference books, containing mathematical formulas.

(*ii*) In the other, generating difficulties case, the analytical expressions are very complicated formulas. Then, if even these solutions are found, they may not be included in many handbooks, for reasons of their extended form. The later property is, in particular, stipulated by the aspiration of the calculators to resolve these integrals in the most general case of the domain of variables.

Dealing with the problems of the types (*i*) and/or (*ii*), we have to overcome these troubles by performing our own computations of the considered integrals. Fortunately, we were able to find an analytical evaluation of an integral, which may serve as an initial starting-point for our further advance. In the reference handbook of Prudnikov et al. [4] is given the following formula for the analytical solution of the integral (formula on page 545; *note that the integration is from* 0 *to* π*, not from* 0 *to* 2π !):

$$\int_0^{\pi}[ln(1-2acos\varphi+a^2)](1-2bcos\varphi+b^2)^{-1}\,d\varphi= \tag{2}$$

$$=\begin{cases} 2\pi\,|1-b^2|^{-1}ln(1-ab^{\pm 1})\,; & \left[\begin{cases}|b|<1\\ |b|>1\end{cases},\ |a|\le 1\right], \\ & \text{or} \\ 2\pi\,|1-b^2|^{-1}ln|a-b^{\pm 1}|\,; & \left[\begin{cases}|b|<1\\ |b|>1\end{cases},\ |a|>1\right]. \end{cases}$$

For further use of the above formula, we shall express the parameters a and b by means of the eccentricity $e(u)$ and its derivative $\dot{e}(u)$ in a way, depending on the kind of the integral, which we intend to evaluate. At first glance, the existence of four possible branches in the right-hand-side of the relation (2), leads to the suspicion that the solutions, which are based on (2), are lacking of uniqueness. We shall see later, that in our applications all the four solutions are, in fact, *identical*. That is to say, the branching in our case makes no sense. We also stress that the eccentricity $e(u)$, its derivative $\dot{e}(u)$ and, correspondingly, their difference $e(u)-\dot{e}(u)$ are real functions of u. In turn, the parameters a and b, expressed in terms of $e(u)$ and $\dot{e}(u)$, are also real quantities. Taking into account that we resolve the task under the conditions of satisfying the inequalities $|e(u)|<1$, $|\dot{e}(u)|<1$ and $|e(u)-\dot{e}(u)|<1$, we could conclude that the integral into the left-hand-side of the equality (2) is a real function on $e(u)$ and $\dot{e}(u)$.Without any singular behaviour in the pointed out domain of these variables.

Our main goal in the present paper is to compute analytically the following two kinds of integrals:

(3) $$\mathbf{L_i}(e,\dot{e}) \equiv \int_0^{2\pi} [ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-i}\, d\varphi \;;\quad \mathbf{i} = 0, \ldots, 3\,,$$

(4) $$\mathbf{K_j}(e,\dot{e}) \equiv \int_0^{2\pi} [ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-j}\, d\varphi \;;\quad \mathbf{j} = 1, \ldots, 5\,.$$

The above integrals resemble to the integrals:

(5) $$\mathbf{A_i}(e,\dot{e}) \equiv \int_0^{2\pi} [1 + (e - \dot{e})cos\varphi]^{-i}\, d\varphi \;;\quad \mathbf{i} = 1, \ldots, 5\,,$$

(6) $$\mathbf{J_j}(e,\dot{e}) \equiv \int_0^{2\pi} (1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-j}\, d\varphi \;;\quad \mathbf{j} = 1, \ldots, 4\,,$$

(7) $$\mathbf{H_j}(e,\dot{e}) \equiv \int_0^{2\pi} (1 + ecos\varphi)^{-j}[1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi \;;\quad \mathbf{j} = 1, \ldots, 4\,,$$

in the sense, that in the denominators of the integrands encounter as factors certain powers of the quantities $(1 + ecos\varphi)$ or $[1 + (e - \dot{e})cos\varphi]$. But for the first system of integrals (3) & (4), the nominators are equal to the logarithmic function $ln(1 + ecos\varphi)$, instead to unity. The later circumstance essentially complicates the analytical evaluations of $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, …, 3) and $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 5), in comparison with the corresponding computations of the integrals $\mathbf{A_i}(e,\dot{e})$, ($\mathbf{i}$ = 1, …, 5), $\mathbf{J_j}(e,\dot{e})$ and $\mathbf{H_j}(e,\dot{e})$, ($\mathbf{j}$ = =1, …, 4), which were done in an earlier paper [4]. Of course, the selection of the powers of the factors $(1 + ecos\varphi)$ and $[1 + (e - \dot{e})cos\varphi]$ into the definitions (3) and (4), is predetermined by the necessity of the applications of the analytical solutions for our own future developments. That is to say, like the situation with $\mathbf{A_i}(e,\dot{e})$, $\mathbf{J_j}(e,\dot{e})$ and $\mathbf{H_j}(e,\dot{e})$, the integrals $\mathbf{L_i}(e,\dot{e})$ and $\mathbf{K_j}(e,\dot{e})$, in principle, may be evaluated analytically for arbitrary non-negative integers $\mathbf{i}$ or $\mathbf{j}$, by the means, which we shall use in the present paper. But we shall limit us further only to the necessary minimum of computations. These will be based on the application of the relation (2), and in connection with this, we make the following *important remark*. We do not trace back the derivation of the formula (2) and accept to trust the adduced solution of the Prudnikov et al. [5]. To preserve us from any possible incorrectness of this formula, we further check the derived analytical expressions also by means of *numerical* computations for a dense enough lattice of values of $e(u)$ and $\dot{e}(u)$. Both in the *open* interval (– 1.0; + 1.0), taking also into account that $|e(u) - \dot{e}(u)| < 1$. Speaking in advance, we note that there is not doubt in the validity of the relation (2), because the discrepancies between the analytical and numerical evaluations (based on the formula (2)) are of the order $10^{-11} - 10^{-13}$ – the accuracy of the numerical computations itself.

2.1. Recurrence relations for the integrals of the type $\mathbf{L_i}(e,\dot{e}) \equiv \int_0^{2\pi} [ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-i} \, d\varphi$

We shall establish in this chapter a number of relations between the integrals given by the definitions (3) and (4), which will enable us to evaluate in an explicit analytical form these integrals as functions of the eccentricity $e(u)$ and its derivative $\dot{e}(u) \equiv de(u)/du$. Actually, these formulas will be recurrence relations for the first kind of integrals, namely, $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, , 3). They will include also integrals of the type $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 5), which, at the present stage of the computations, *are still unknown* functions of $e(u)$ and $\dot{e}(u)$. Later we shall find another recurrence relations about $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 5), that refer *only* to this kind of integrals. As a final result, this will give us an opportunity to calculate in an explicit form the integrals $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 5). Returning back to the recurrence relations for $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, …, 3), derived below in the present chapter, we finally are in a position to write explicitly the analytical solutions for the integrals $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, …, 3).

We begin with the remark that $\mathbf{L_0}(e)$ is a function only of the eccentricity $e(u)$, but not of its derivative $\dot{e}(u)$:

$$(8) \qquad \mathbf{L_0}(e) \equiv \int_0^{2\pi} [ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1} \, d\varphi.$$

To the end of the present chapter, we shall suppose that $e(u) \neq 0$ and $\dot{e}(u) \neq 0$. The evaluations of $\mathbf{L_1}(e,\dot{e})$, $\mathbf{L_2}(e,\dot{e})$ and $\mathbf{L_3}(e,\dot{e})$ for these particular values of their arguments are more appropriate to be given, when the full expressions for them are already available. According to the definitions (3) and (4), we have:

$$(9) \qquad \mathbf{L_3}(e,\dot{e}) \equiv \int_0^{2\pi} [ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-3} \, d\varphi =$$

$$= \int_0^{2\pi} \{[1 + (e - \dot{e})cos\varphi] - (e - \dot{e})cos\varphi\}[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1} \times$$

$$\times [1 + (e - \dot{e})cos\varphi]^{-3} \, d\varphi = \int_0^{2\pi} [ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-2} \, d\varphi -$$

$$- [(e - \dot{e})/e]\int_0^{2\pi}[(1 + e\cos\varphi) - 1][\ln(1 + e\cos\varphi)](1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-3}\, d\varphi =$$

$$= \mathbf{L_2}(e,\dot{e}) - [(e - \dot{e})/e]\int_0^{2\pi}[\ln(1 + e\cos\varphi)][1 + (e - \dot{e})\cos\varphi]^{-3}\, d\varphi +$$

$$+ [(e - \dot{e})/e]\int_0^{2\pi}[\ln(1 + e\cos\varphi)](1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-3}\, d\varphi =$$

$$= \mathbf{L_2}(e,\dot{e}) - [(e - \dot{e})/e]\mathbf{K_3}(e,\dot{e}) + [(e - \dot{e})/e]\mathbf{L_3}(e,\dot{e}).$$

Therefore, transferring the last term $[(e - \dot{e})/e]\mathbf{L_3}(e,\dot{e})$ into the right-hand-side, we obtain that:

(10) $\{1 - [(e - \dot{e})/e]\}\mathbf{L_3}(e,\dot{e}) \equiv (\dot{e}/e)\mathbf{L_3}(e,\dot{e}) = \mathbf{L_2}(e,\dot{e}) - [(e - \dot{e})/e]\mathbf{K_3}(e,\dot{e})$,

or, multiplying by $e/\dot{e}$:

(11) $\mathbf{L_3}(e,\dot{e}) = (e/\dot{e})\mathbf{L_2}(e,\dot{e}) - [(e - \dot{e})/\dot{e}]\mathbf{K_3}(e,\dot{e})$.

By the exactly analogous way, we may derive recurrence relations for the integrals $\mathbf{L_2}(e,\dot{e})$ and $\mathbf{L_1}(e,\dot{e})$. We simply write here the final results:

(12) $$\mathbf{L_2}(e,\dot{e}) \equiv \int_0^{2\pi}[\ln(1 + e\cos\varphi)](1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-2}\, d\varphi =$$

$$= (e/\dot{e})\mathbf{L_1}(e,\dot{e}) - [(e - \dot{e})/\dot{e}]\mathbf{K_2}(e,\dot{e}),$$

(13) $$\mathbf{L_1}(e,\dot{e}) \equiv \int_0^{2\pi}[\ln(1 + e\cos\varphi)](1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi =$$

$$= (e/\dot{e})\mathbf{L_0}(e,\dot{e}) - [(e - \dot{e})/\dot{e}]\mathbf{K_1}(e,\dot{e}).$$

We call to mind, that the above formulas are deduced under the assumptions that $e(u) \neq 0$ and $\dot{e}(u) \neq 0$. Obviously, the equations (11), (12) and (13) are useful if $\mathbf{K_1}(e,\dot{e})$, $\mathbf{K_2}(e,\dot{e})$ and $\mathbf{K_3}(e,\dot{e})$ are already known functions of $e(u)$ and $\dot{e}(u)$.

2.2. Analytical computation of the integral

$$\mathbf{L_0}(e) \equiv \int_0^{2\pi}[\ln(1 + e\cos\varphi)](1 + e\cos\varphi)^{-1}\, d\varphi$$

For our present purposes we shall transform the left-hand-side of the formula (2) (given by Prudnikov et al. [5]) in the following way:

(14) $$\int_0^{\pi}[\ln(1 - 2a\cos\varphi + a^2)](1 - 2b\cos\varphi + b^2)^{-1}\, d\varphi =$$

$$= \int_0^{\pi} \{ln\{(1 + a^2)\{1 - [2a/(1 + a^2)]cos\varphi\}\}\}(1 + b^2)^{-1}\{1 - [2b/(1 + b^2)]cos\varphi\}^{-1} d\varphi =$$

$$= (1 + b^2)^{-1}[ln(1 + a^2)]\int_0^{\pi}\{1 - [2b/(1 + b^2)]cos\varphi\}^{-1} d\varphi +$$

$$+ (1 + b^2)^{-1}\int_0^{\pi} \{ln\{1 - [2a/(1 + a^2)]cos\varphi\}\}\{1 - [2b/(1 + b^2)]cos\varphi\}^{-1} d\varphi.$$

We remind here that, unlike the previous integrals, now *the integration is from* 0 *to π, not from* 0 *to* 2*π* ! Let us substitute:

(15) $-[2a/(1 + a^2)] = -[2b/(1 + b^2)] = e(u)$.

Therefore, we are able to write, in agreement with the relation (2), that:

(16) $$\int_0^{\pi}[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1} d\varphi = -\pi[ln(1 + a^2)](1 - e^2)^{-1/2} +$$

$$+ \begin{cases} 2\pi\,(1 + b^2)|1 - b^2|^{-1}ln(1 - ab^{\pm 1})\,; \quad \left[\begin{cases} |b| < 1 \\ |b| > 1 \end{cases},\ |a| \le 1\right], \quad \text{or} \\[2ex] 2\pi\,(1 + b^2)|1 - b^2|^{-1}ln|a - b^{\pm 1}|\,; \quad \left[\begin{cases} |b| < 1 \\ |b| > 1 \end{cases},\ |a| > 1\right]. \end{cases}$$

We shall now check the validity of the "obvious" condition $a = b$. We have that:

(17) $-[2a/(1 + a^2)] = -[2b/(1 + b^2)] \Rightarrow a + ab^2 = b + a^2b$, or

(18) $ab^2 - a^2b - b + a = 0$.

We write the above equality as a quadratic equation with respect to b:

(19) $ab^2 + (-1 - a^2)b + a = 0$.

Correspondingly, the solutions of this equation are:

(20) $b_{I,II} = [a^2 + 1 \pm (1 + 2a^2 + a^4 - 4a^2)^{1/2}]/(2a) = \{a^2 + 1 \pm [(1 - a^2)^2]^{1/2}\}/(2a) =$
$= [a^2 + 1 \pm (1 - a^2)]/(2a)$.

The two solutions are, therefore:

(21) $b_I = (a^2 + 1 - 1 + a^2)/(2a) \equiv a$, and

(22) $b_{II} = (a^2 + 1 + 1 - a^2)/(2a) \equiv 1/a$.

The existence of the above equalities means that we *have to consider two cases*: $b_I = a$ and $b_{II} = 1/a$. However, let us examine at first, how the restriction $|e(u)| < 1$ imposes itself other restrictions over the variables a and b. We have that, according to the substitution (15):

(23) $-2a = e + ea^2$,

which implies a quadratic equation for a:

(24) $ea^2 + 2a + e = 0$.

Solving with respect to a this quadratic equation, we obtain:

(25) $a_{1,2} = [-2 \pm (4 - 4e^2)^{1/2}]/(2e) = [-1 \pm (1 - e^2)^{1/2}]/e$.

We have to investigate the following *four* situations:

(*i*) ***solution*** $\boldsymbol{a_1} = [-1 + (1 - e^2)^{1/2}]/e$; $\boldsymbol{|a_1| = |-1 + (1 - e^2)^{1/2}|/|e| < 1}$. This implies that:

(26) $|-1 + (1 - e^2)^{1/2}| < |e|$.

Because $|e| < 1$, we can write $0 < 1 - e^2 < 1$, or $(1 - e^2)^{1/2} < 1$.
Further we have:
$-1 + (1 - e^2)^{1/2} < 0$, which means that $|-1 + (1 - e^2)^{1/2}| = 1 - (1 - e^2)^{1/2}$.
Consequently, from the inequality (26) follows that: $0 < 1 - (1 - e^2)^{1/2} < |e|$.
Therefore:
$0 < 1 - |e| < (1 - e^2)^{1/2}$. Raising into square will give:

(27) $1 + e^2 - 2|e| < 1 - e^2 \;\Rightarrow\; 2e^2 < 2|e| \;\Rightarrow\; e^2 < |e| \;\Rightarrow |e| < 1$.

The above chain of inequalities means that we do not arrive at a contradiction. That is to say, this case (*i*) is admissible.

(*ii*) ***solution*** $\boldsymbol{a_1} = [-1 + (1 - e^2)^{1/2}]/e$; $\boldsymbol{|a_1| = |-1 + (1 - e^2)^{1/2}|/|e| > 1}$.This implies that:

(28) $|-1 + (1 - e^2)^{1/2}| > |e|$.

Because $|e| < 1$, we can write $e^2 < 1$, or $0 < 1 - e^2 < 1$, and $(1 - e^2)^{1/2} < 1$. Further we have: $-1 + (1 - e^2)^{1/2} < 0$, which means that $|-1 + (1 - e^2)^{1/2}| = 1 - (1 - e^2)^{1/2}$. We shall substitute the later equality into the inequality (28). Unlike the previous case (*i*), now the sign of the inequality (28) is changed in comparison with (26). This will introduce a radical change in our conclusions. According to (28), we write $[0 < 1 - (1 - e^2)^{1/2}] \cap [1 - (1 - e^2)^{1/2} > |e|]$, or $[0 < 1 - |e|] \cap [1 - |e| > (1 - e^2)^{1/2}]$. Raising into square will give:

(29) $1 + e^2 - 2|e| > 1 - e^2 \;\Rightarrow\; 2e^2 > 2|e| \;\Rightarrow\; e^2 > |e| \;\Rightarrow\; |e| > 1$.

We derive a contradiction, because, by hypothesis, $|e| < 1$. That is to say, the considered case (*ii*) is not permitted!

(*iii*) ***solution*** $\boldsymbol{a_2} = [-1-(1-e^2)^{1/2}]/e$; $\boldsymbol{|a_2| = |-1-(1-e^2)^{1/2}|/|e| < 1}$.This implies that:

(30) $|-1-(1-e^2)^{1/2}| \equiv 1+(1-e^2)^{1/2} < |e|$.

This relation may be immediately rewritten as:

(31) $0 < 1-|e| < -(1-e^2)^{1/2} < 0.$

It turns out that must be fulfilled simultaneously *two* inequalities about the difference $1-|e|$:

(33) $(1-|e| > 0) \cap (1-|e| < 0).$

i.e., we obtain a contradiction. Therefore, the considered case (*iii*) is not permitted!

(*iv*) ***solution*** $\boldsymbol{a_2} = [-1-(1-e^2)^{1/2}]/e$; $\boldsymbol{|a_2| = |-1-(1-e^2)^{1/2}|/|e| > 1}$.This implies that:

(34) $|-1-(1-e^2)^{1/2}| \equiv 1+(1-e^2)^{1/2} > |e|$,

(35) $1-|e| > -(1-e^2)^{1/2} \;\Rightarrow\; 1-|e| > 0 > -(1-e^2)^{1/2}.$

Correspondingly, in this case (*iv*) we do not strike with a contradiction.

To summarize the conclusions from the considered above four possible opportunities, we shortly say that, under the restriction $|e(u)| < 1$:

1) The solution of the quadratic equation (24) $a_1 = [-1+(1-e^2)^{1/2}]/e$ is in agreement with this restriction only if $\boldsymbol{|a| < 1}$;

2) The solution of the quadratic equation (24) $a_2 = [-1-(1-e^2)^{1/2}]/e$ is in agreement with this restriction only if $\boldsymbol{|a| > 1}$.

The situation is illustrated graphically in Figure 1 (*a*) and (*b*).

Because of the symmetry, given by the equalities (15), the same conclusions are valid for the coefficient b, where we have to consider the quadratic equation

(36) $eb^2 + 2b + e = 0,$

instead of the equation (24).

3) The solution of the quadratic equation (36) $b_1 = [-1+(1-e^2)^{1/2}]/e$ is in agreement with the restriction $|e(u)| < 1$ only if $\boldsymbol{|b| < 1}$;

4) The solution of the quadratic equation (36) $b_2 = [-1-(1-e^2)^{1/2}]/e$ is in agreement with the restriction $|e(u)| < 1$ only if $\boldsymbol{|b| > 1}$.

We must not confuse the roots $\boldsymbol{b_{I}}$ and $\boldsymbol{b_{II}}$ with the roots $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$!

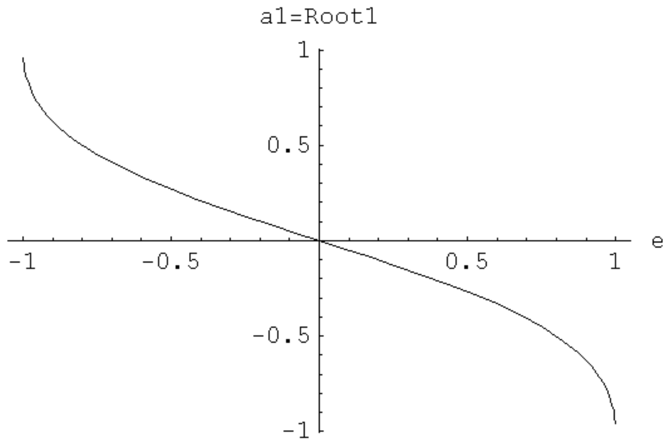
Having in mind the above preliminary remarks, we now return to the investigation of the equation (19).

Case I: *b* = *a*. Subcase 1: |*b*| = |*a*| < 1.

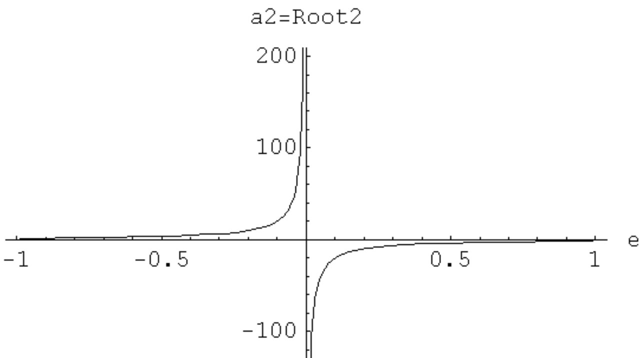
According to the deductions 1) and 3), $a_1 = b_1 = [-1 + (1 - e^2)^{1/2}]/e$. Then, in view of the formula (16):

(37) $$\int_0^{\pi}[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}\,d\varphi = -\pi\{ln[1 + (a_1)^2]\}(1 - e^2)^{-1/2} +$$

$$+ 2\pi[1 + (a_1)^2][1 - (a_1)^2]^{-1}ln[1 - (a_1)^2].$$



a) Case (i): $|a_1| = |-1 + (1 - e^2)^{1/2}|/|e| < 1$.



b) Case (iv): $|a_2| = |-1 - (1 - e^2)^{1/2}|/|e| > 1$.

Fig.1. Solutions of the quadratic equation (24) $ea^2 + 2a + e = 0$

We compute the following auxiliary expressions:

(38) $1 + (a_1)^2 = 1 + (1/e^2)[1 + 1 - e^2 - 2(1 - e^2)^{1/2}] = (2/e^2)[1 - (1 - e^2)^{1/2}] > 0,$

(39) $1 - (a_1)^2 = 1 - (1/e^2)[1 + 1 - e^2 - 2(1 - e^2)^{1/2}] = (1/e^2)[e^2 - 2 + e^2 + 2(1 - e^2)^{1/2}] =$

$= (2/e^2)[e^2 - 1 + (1 - e^2)^{1/2}][1 + (1 - e^2)^{1/2}][1 + (1 - e^2)^{1/2}]^{-1} =$

$= 2e^2(1 - e^2)^{1/2}/\{e^2[1 + (1 - e^2)^{1/2}]\} \equiv 2(1 - e^2)^{1/2}/[1 + (1 - e^2)^{1/2}].$

Consequently, we have:

(40) $0 < 1 - (a_1)^2 = 2(1 - e^2)^{1/2}/[1 + (1 - e^2)^{1/2}] < 1.$

(41) $\int_0^\pi [ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}\, d\varphi =$ ***solution 1*** $\equiv$

$\equiv - \pi(1 - e^2)^{-1/2} ln\{(1/e^2)[2 - 2(1 - e^2)^{1/2}]\} +$
$+ 2\pi(2/e^2)[1 - (1 - e^2)^{1/2}][1 + (1 - e^2)^{1/2}][2(1 - e^2)^{1/2}]^{-1}\times$
$\times ln\{2(1 - e^2)^{1/2}/[1 + (1 - e^2)^{1/2}]\} = - \pi(1 - e^2)^{-1/2} ln\{(1/e^2)[2 - 2(1 - e^2)^{1/2}]\} +$
$+ 2\pi(1 - e^2)^{-1/2} ln\{2(1 - e^2)^{1/2}/[1 + (1 - e^2)^{1/2}]\} = \pi(1 - e^2)^{-1/2} ln\{4e^2(1 - e^2)\times$
$\times\{2[1 - (1 - e^2)^{1/2}][1 + (1 - e^2)^{1/2}][1 + (1 - e^2)^{1/2}]\}^{-1}\} =$
$= \pi(1 - e^2)^{-1/2} ln\{2e^2(1 - e^2)\{(1 - 1 + e^2)[1 + (1 - e^2)^{1/2}]\}^{-1}\}.$

Finally, we are able to write for this case:

(42) $\int_0^\pi [ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}\, d\varphi =$ ***solution 1*** $\equiv$

$\equiv \pi(1 - e^2)^{-1/2} ln\{2(1 - e^2)[1 + (1 - e^2)^{1/2}]^{-1}\} \equiv$
$\equiv - \pi(1 - e^2)^{-1/2} ln\{[1 + (1 - e^2)^{1/2}][2(1 - e^2)]^{-1}\}.$

Case I: *b* = *a*. Subcase 2: |*b*| = |*a*| > 1.

According to the deductions 2) and 4), $a_2 = b_2 = [-1 - (1 - e^2)^{1/2}]/e$. Then, in view of the formula (16):

(43) $\int_0^\pi [ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}\, d\varphi = - \pi(1 - e^2)^{-1/2} ln[1 + (a_2)^2] +$

$+ 2\pi[1 + (a_2)^2]|1 - (a_2)^2|^{-1} ln|a_2 - 1/a_2|.$

We compute the following auxiliary expressions:

(44) $1 + (a_2)^2 = 1 + (1/e^2)[1 + 1 - e^2 + 2(1 - e^2)^{1/2}] = (2/e^2)[1 + (1 - e^2)^{1/2}],$

(45) $1 - (a_2)^2 = 1 - (1/e^2)[1 + 1 - e^2 + 2(1 - e^2)^{1/2}] = (1/e^2)[e^2 - 2 + e^2 - 2(1 - e^2)^{1/2}] =$

$= (2/e^2)[e^2 - 1 - (1 - e^2)^{1/2}][1 - (1 - e^2)^{1/2}][1 - (1 - e^2)^{1/2}]^{-1} =$

$= - 2(1 - e^2)^{1/2}[1 - (1 - e^2)^{1/2}]^{-1} < 0,$

(46) $|a_2 - 1/a_2| = |[(a_2)^2 - 1]/a_2| = |(a_2)^2 - 1|/|a_2| = [(a_2)^2 - 1]/|a_2| =$

$= 2|e|(1 - e^2)^{1/2}[1 + (1 - e^2)^{1/2}]^{-1}[1 - (1 - e^2)^{1/2}]^{-1} =$

$= 2|e|(1 - e^2)^{1/2}(1 - 1 + e^2)^{-1} = 2(1 - e^2)^{1/2}/|e|.$

The derivation of the equality (46) takes into account the last inequality in (45), according to which $(a_2)^2 - 1 > 0$, implying that $|(a_2)^2 - 1| = (a_2)^2 - 1$. Therefore, in the present subcase we can write:

$$(47)\quad \int_0^{\pi}[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}\, d\varphi = \textbf{solution2} \equiv$$

$$\equiv - \pi(1 - e^2)^{-1/2}ln\{(2/e^2)[1 + (1 - e^2)^{1/2}]\} + \pi(1 - e^2)^{-1/2}ln[4(1 - e^2)/e^2] =$$
$$= - \pi(1 - e^2)^{-1/2}ln\{2e^2[1 + (1 - e^2)^{1/2}]/[4e^2(1 - e^2)]\} =$$
$$= - \pi(1 - e^2)^{-1/2}ln\{[1 + (1 - e^2)^{1/2}][2(1 - e^2)]^{-1}\} = \textbf{solution 1}.$$

We see that the both subcases give the same analytical solutions for the integral that we are resolving!

Case II: $b = 1/a$. Subcase 3: $|b| = |1/a| < 1$.

According to the deductions 2) and 3), $a = a_2 = [-1 - (1 - e^2)^{1/2}]/e$ and $b = b_1 = [-1 + (1 - e^2)^{1/2}]/e$. Then, applying formula (16):

$$(48)\quad \int_0^{\pi}[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}\, d\varphi = \textbf{solution 3} \equiv - \pi(1 - e^2)^{-1/2}ln[1 + (a_2)^2] +$$
$$+ 2\pi[1 + (b_1)^2]|1 - (b_1)^2|^{-1}ln|a_2 - b_1| .$$

Note that $b_1 = (1/e)[-1 + (1 - e^2)^{1/2}][1 + (1 - e^2)^{1/2}][1 + (1 - e^2)^{1/2}]^{-1} = (1/e)(1 - e^2 - 1)[1 + (1 - e^2)^{1/2}]^{-1} = e[-1 - (1 - e^2)^{1/2}]^{-1} = 1/(a_2)$, i.e., $b_1 = 1/(a_2)$. This result once again affirms the consistency of our calculations. We shall use the already computed expressions (38) and (39) for $1 + (a_1)^2$ and $1 - (a_1)^2$, respectively, because

$1 + (b_1)^2 = 1 + (a_1)^2$ and $|1 - (b_1)^2| = |1 - (a_1)^2|$.

Moreover:

$$(49)\quad |a_2 - b_1| = |a_2 - 1/(a_2)|.$$

After these remarks, we have:

$$(50)\quad \int_0^{\pi}[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}\, d\varphi = \textbf{solution 3} \equiv$$

$$= - \pi(1 - e^2)^{-1/2}ln\{(2/e^2)[1 + (1 - e^2)^{1/2}]\} + 2\pi\{2[1 - (1 - e^2)^{1/2}][1 + (1 - e^2)^{1/2}]\times$$
$$\times[2e^2(1 - e^2)^{1/2}]^{-1}ln[2(1 - e^2)^{1/2}|e|^{-1}] = - \pi(1 - e^2)^{-1/2}ln\{(2/e^2)[1 + (1 - e^2)^{1/2}]\} +$$
$$+ \pi(1 - e^2)^{-1/2}[(1 - 1 + e^2)/e^2]ln[4(1 - e^2)/e^2] =$$
$$= - \pi(1 - e^2)^{-1/2}ln\{[1 + (1 - e^2)^{1/2}][2(1 - e^2)]^{-1}\} = \textbf{solution 1}.$$

We again have a coincidence with the earlier evaluations of the considered integral.

Let us to proceed to the last remaining case in formula (16).

Case II: $b = 1/a$. Subcase 4: $|b| = 1/|a| > 1$.

According to the deductions 1) and 4), we have $a = a_1 = (1/e)[-1 + (1-e^2)^{1/2}]$ and $b = b_2 = (1/e)[-1-(1-e^2)^{1/2}]$. Note that $b_2 = (1/e)[-1 - (1-e^2)^{1/2}][1-(1-e^2)^{1/2}]\times[1-(1-e^2)^{1/2}]^{-1} = (-1/e)(1-1+e^2)[1-(1-e^2)^{1/2}]^{-1} = e[-1+(1-e^2)^{1/2}]^{-1} = 1/a_1$, i.e., $b_2 = 1/a_1$. This result once again affirms the consistency of our calculations.

(51) $$\int_0^{\pi}[ln(1+ecos\varphi)](1+ecos\varphi)^{-1}\,d\varphi = \textbf{\textit{solution 4}} \equiv -\pi(1-e^2)^{-1/2}ln[1+(a_1)^2] +$$
$$+\,2\pi[1+(b_2)^2]|1-(b_2)^2|^{-1}ln|1-a_1/b_2|\,.$$

Taking into account that $1+(b_2)^2 = 1+(a_2)^2$ (see equality (44)) and $|1-(b_2)^2| = |1-(a_2)^2| = 2(1-e^2)^{1/2}[1-(1-e^2)^{1/2}]^{-1}$ (see equality (45)), and also $(1-a_1/b_2) = [1-(a_1)^2] = 2(1-e^2)^{1/2}[1+(1-e^2)^{1/2}]^{-1}$, we find that in this subcase:

(52) $$\int_0^{\pi}[ln(1+ecos\varphi)](1+ecos\varphi)^{-1}\,d\varphi = \textbf{\textit{solution 4}} \equiv$$
$$\equiv -\pi(1-e^2)^{-1/2}ln\{(2/e^2)[1-(1-e^2)^{1/2}]\} + 2\pi\{2[1+(1-e^2)^{1/2}][1-(1-e^2)^{1/2}]\times$$
$$\times[2e^2(1-e^2)^{1/2}]^{-1}ln\{2(1-e^2)^{1/2}[1+(1-e^2)^{1/2}]^{-1}\} =$$
$$= -\pi(1-e^2)^{-1/2}ln\{(2/e^2)[1-(1-e^2)^{1/2}]\} +$$
$$+\,\pi(1-e^2)^{-1/2}[(1-1+e^2)/e^2]ln\{4(1-e^2)[1+(1-e^2)^{1/2}]^{-2}\} =$$
$$= -\pi(1-e^2)^{-1/2}ln\{2[1-(1-e^2)^{1/2}][1+(1-e^2)^{1/2}][1+(1-e^2)^{1/2}][4e^2(1-e^2)]^{-1}\}=$$
$$= -\pi(1-e^2)^{-1/2}ln\{(1-1+e^2)[1+(1-e^2)^{1/2}][2e^2(1-e^2)]^{-1}\} =$$
$$= -\pi(1-e^2)^{-1/2}ln\{[1+(1-e^2)^{1/2}][2(1-e^2)]^{-1}\} = \textbf{\textit{solution 1}}.$$

The final conclusion is that for all cases/subcases we obtain identical results. It is also easily checked, that the extension of the interval of integration over the azimuthal angle φ from $[0, \pi]$ to $[0, 2\pi]$, simply leads to a multiplying of the results by a factor of two. Therefore:

(53) $$\mathbf{L_0}(e) \equiv \int_0^{2\pi}[ln(1+ecos\varphi)](1+ecos\varphi)^{-1}\,d\varphi =$$
$$= -2\pi(1-e^2)^{-1/2}ln\{[1+(1-e^2)^{1/2}][2(1-e^2)]^{-1}\}.$$

It must be emphasized that the above derivations are performed under the condition $e(u) \neq 0$. Bur the direct computation for $\mathbf{L_0}(0)$ gives **a** zero value, because for this case $ln(1 + ecos\varphi) \equiv 0$. The same evaluation follows also from the formula (53), though it was established under non-zero values of the eccentricity $e(u)$. Consequently, we are able to apply the evaluation (53) for $\mathbf{L_0}(e)$ for arbitrary values of $e(u)$ belonging to the open interval (– 1.0; + 1.0).

2. 3. Analytical computation of the integral

$$\mathbf{K_1}(e,\dot{e}) \equiv \int_0^{2\pi} [ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi$$

First of all, we note that the integral $\mathbf{K_0}(e)$, which does not depend on the derivative $\dot{e}(u) \equiv de(u)/du$, is already calculated in [6]. According to formula **865.44** in this handbook of formulas, we have:

(54) $$\mathbf{K_0}(e) \equiv \int_0^{2\pi} ln(1 + ecos\varphi)\, d\varphi = 2\pi\, ln\{[1 + (1 - e^2)^{1/2}]/2\}.$$

Our main purpose in the present chapter is to find analytical evaluations for the integrals $\mathbf{K_i}(e,\dot{e})$, (i = 1, …, 5), given by the definition (4). For this reason, we rewrite formula (2) into the following, more suitable form (see also the relation (14)):

(55) $$\int_0^{\pi} [ln(1 - 2acos\varphi + a^2)](1 - 2bcos\varphi + b^2)^{-1}\, d\varphi =$$

$$= (1 + b^2)^{-1}[ln(1 + a^2)]\int_0^{\pi} \{1 - [2b/(1 + b^2)]cos\varphi\}^{-1}\, d\varphi +$$

$$+ (1 + b^2)^{-1}\int_0^{\pi} \{ln\{1 - [2a/(1 + a^2)]cos\varphi\}\}\{1 - [2b/(1 + b^2)]cos\varphi\}^{-1}\, d\varphi =$$

$$= (1 + b^2)^{-1}[ln(1 + a^2)]\int_0^{\pi} [1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi +$$

$$+ (1 + b^2)^{-1}\int_0^{\pi} [ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi.$$

We have used above the two substitutions:

(56) $-2a/(1 + a^2) = e(u)$, and

(57) $-2b/(1 + b^2) = e(u) - \dot{e}(u)$.

We use also the result/formula **858.524** from Dwight [6]:

(58) $$\int_0^{\pi} [1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi = \pi[1 - (e - \dot{e})^2]^{-1/2}.$$

Therefore:

(59) $$\int_0^{\pi} [ln(1 - 2acos\varphi + a^2)](1 - 2bcos\varphi + b^2)^{-1}\, d\varphi =$$

$$= -\pi[1 - (e - \dot{e})^2]^{-1/2} ln(1 + a^2) +$$

$$
+\begin{cases}
2\pi (1 + b^2)|1 - b^2|^{-1} ln(1 - ab^{\pm 1}) \; ; & \left[\begin{cases} |b| < 1 \\ |b| > 1 \end{cases} , \; |a| \le 1 \right] , \\
 & \qquad\qquad\qquad\qquad\qquad\qquad \text{or} \\
2\pi (1 + b^2)|1 - b^2|^{-1} ln|a - b^{\pm 1}| \; ; & \left[\begin{cases} |b| < 1 \\ |b| > 1 \end{cases} , \; |a| > 1 \right] .
\end{cases}
$$

The solution of the equation (56) gives two roots:

(60) $a_1 = [-1 + (1 - e^2)^{1/2}]/e,$

(61) $a_2 = [-1 - (1 - e^2)^{1/2}]/e.$

The solution of the equation (57) also gives two roots:

(62) $b_1 = \{-1 + [1 - (e - \dot{e})^2]^{1/2}\}/(e - \dot{e}),$

(63) $b_2 = \{-1 - [1 - (e - \dot{e})^2]^{1/2}\}/(e - \dot{e}).$

The restriction $|e(u)| < 1$ implies that $|a_1| < 1$ and $|a_2| > 1$. From the other hand, the restriction $|e(u) - \dot{e}(u)| < 1$ implies that $|b_1| < 1$ and $|b_2| > 1$. Let us find relations between the systems of roots $\{a_1, a_2\}$ and $\{b_1, b_2\}$, respectively. From substitutions (56) and (57) follows that:

(64) $-2b/(1 + b^2) + 2a/(1 + a^2) = -\dot{e}(u).$

Multiplication of this equality by $(1 + a^2)(1 + b^2)$ leads to a new form of this relation:

(65) $-2b(1 + a^2) + 2a(1 + b^2) = -\dot{e}(1 + a^2)(1 + b^2) \quad \Leftrightarrow$

$\Leftrightarrow \quad -2b - 2ba^2 + 2a + 2ab^2 + \dot{e} + \dot{e}b^2 + \dot{e}a^2 + \dot{e}a^2b^2 = 0.$

If we consider the variable b as an unknown quantity, the later equality may be regarded as a quadratic equation for b:

(66) $(\dot{e}a^2 + 2a + \dot{e})b^2 + (-2 - 2a^2)b + (\dot{e}a^2 + 2a + \dot{e}) = 0.$

Taking into account the equality (56), we compute that:

(67) $\dot{e}a^2 + 2a + \dot{e} = 2a + \dot{e}(1 + a^2) = 2a + \dot{e}(-2a/e) = 2a(1 - \dot{e}/e) = (2a/e)(e - \dot{e}).$

Moreover:

(68) $-2 - 2a^2 = -2(1 + a^2) = 4a/e.$

Therefore, the quadratic equation (66) becomes (after dividing by $2a/e$):

(69) $(e - \dot{e})b^2 + 2b + (e - \dot{e}) = 0.$

The two roots of this equation are:

(70) $b_1 = \{-1 + [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-1}$,

(71) $b_2 = \{-1 - [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-1}$.

In the opposite case, we also may consider the equation (65) as a quadratic equation for the unknown variable a. Then we obtain the following quadratic equation:

(72) $(\dot{e}b^2 - 2b + \dot{e})a^2 + (2 + 2b^2)a + (\dot{e}b^2 - 2b + \dot{e}) = 0$.

We compute that (in view of the equality (57)):

(73) $\dot{e}b^2 - 2b + \dot{e} = -2b + \dot{e}(1 + b^2) = -2b + \dot{e}[-2b/(e - \dot{e})] =$

$= [-2b/(e - \dot{e})](\dot{e} + e - \dot{e}) = -2be/(e - \dot{e})$.

Moreover:

(74) $2 + 2b^2 = 2(1 + b^2) = -4b/(e - \dot{e})$.

Then, the quadratic equation (72) becomes (after dividing by $-2b/(e - \dot{e})$):

(75) $ea^2 + 2a + e = 0$.

The two roots of this equation are:

(76) $a_1 = [-1 + (1 - e^2)^{1/2}]/e$,

(77) $a_2 = [-1 - (1 - e^2)^{1/2}]/e$.

We stress that the equations (24) and (75) are identical, and, correspondingly, their roots (25) and {(76), (77)} coincide. But the situation is different when we compare the quadratic equations (36) and (69). Because, generally speaking, $\dot{e}(u) \neq 0$, we have not coincidence between these relations, and, consequently, their solutions *are not identical*. For this reason, the notations $\{b_1, b_2\}$ in the present chapter must not be confused with the corresponding notations for the roots in the preceding chapter! With this remark, we continue our investigation of the (possible) relations between the two systems of roots $\{a_1, a_2\}$ and $\{b_1, b_2\}$. In general, the solutions of the equations (69) and (75) imply that we have four self-consistent representations of the analytical expression for the integral

$\mathbf{K_1}(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + e\cos\varphi)][1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi$. Namely: $\{a_1, b_1\}$, $\{a_1, b_2\}$, $\{a_2, b_1\}$ and $\{a_2, b_2\}$. We shall prove now that all these four solutions for $\mathbf{K_1}(\boldsymbol{e},\dot{\boldsymbol{e}})$ are equivalent! In view of this purpose, we consider the following four cases:

Case I: $\boldsymbol{a_1 = [-1 + (1 - e^2)^{1/2}]/e,\ b_1 = \{-1 + [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-1}}$**.**

For this case $|a_1| < 1$ and $|b_1| < 1$. We have already computed that $1 + (a_1)^2 = (2/e^2)[1 - (1 - e^2)^{1/2}] > 0$ (see equality (38)). Further we calculate that:

$$(78)\qquad 1 + (b_1)^2 = 1 + \{1 + 1 - (e - \dot{e})^2 - 2[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2} =$$
$$= 2\{1 - [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2}.$$

The next evaluation is:

$$(79)\qquad |1 - (b_1)^2| = 1 - (b_1)^2 = 1 - \{1 + 1 - (e - \dot{e})^2 - 2[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2} =$$
$$= \{(e - \dot{e})^2 - 2 + (e - \dot{e})^2 + 2[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2} =$$
$$= 2\{(e - \dot{e})^2 + [1 - (e - \dot{e})^2]^{1/2} - 1\}(e - \dot{e})^{-2}.$$

Further we have:

$$(80)\qquad 1 - a_1 b_1 = 1 - [e(e - \dot{e})]^{-1}[-1 + (1 - e^2)^{1/2}]\{-1 + [1 - (e - \dot{e})^2]^{1/2}\} =$$
$$= [e(e - \dot{e})]^{-1}\{e(e - \dot{e}) - \{1 - [1 - (e - \dot{e})^2]^{1/2} - (1 - e^2)^{1/2} +$$
$$+ (1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}\}.$$

Having available the above preliminary evaluations, we can write (in accordance to the formula (59)):

$$(81)\qquad (1/2)\mathbf{K_1}(e,\dot{e}) \equiv \int_0^{\pi}[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi = \boldsymbol{solution\ a_1 b_1} =$$
$$= -\pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(2/e^2)[1 - (1 - e^2)^{1/2}]\} - 2\pi[2/(e - \dot{e})^2]\times$$
$$\times\{1 - [1 - (e - \dot{e})^2]^{1/2}\}[(e - \dot{e})^2/2]\{1 - [1 - (e - \dot{e})^2]^{1/2} - (e - \dot{e})^2\}^{-1}\times$$
$$\times ln\{\{e(e - \dot{e}) - 1 + (1 - e^2)^{1/2} + [1 - (e - \dot{e})^2)]^{1/2} - (1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}\times$$
$$\times[e(e - \dot{e})]^{-1}\} =$$
$$= -\pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(2/e^2)[1 - (1 - e^2)^{1/2}]\} - \pi\{1 - [1 - (e - \dot{e})^2]^{1/2}\}\times$$
$$\times\{1 - [1 - (e - \dot{e})^2]^{1/2} - (e - \dot{e})^2\}^{-1} ln\{\{e^2(e^2 + \dot{e}^2 - 2e\dot{e}) + 1 + 1 - e^2 + 1 -$$
$$- (e^2 + \dot{e}^2 - 2e\dot{e}) + (1 - e^2)(1 - e^2 - \dot{e}^2 + 2e\dot{e}) - 2(e^2 - e\dot{e}) + 2(e^2 - e\dot{e})(1 - e^2)^{1/2} +$$
$$+ 2(e^2 - e\dot{e})[1 - (e - \dot{e})^2)]^{1/2} - 2(e^2 - e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2)]^{1/2} - 2(1 - e^2)^{1/2} -$$
$$- 2[1 - (e - \dot{e})^2)]^{1/2} + 2(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2} + 2(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2} -$$
$$- 2[1 - (e - \dot{e})^2]^{1/2} + 2e^2[1 - (e - \dot{e})^2]^{1/2} - 2(1 - e^2)^{1/2} -$$
$$- \quad 2(1 - e^2)^{1/2}(- e^2 - \dot{e}^2 + 2e\dot{e})\}[e^2(e - \dot{e})^2]^{-1}\}.$$

In the above derivation we have taken into account that the transition of the integration over the azimuthal angle φ from the interval $[0, \pi]$ to the interval $[0, 2\pi]$ gives exactly doubling of the result. That is to say:

$$(82)\qquad \mathbf{K_1}(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi =$$
$$= 2\int_0^{\pi}[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi.$$

We also note that we shall use further the following equality:

(83) $[1-(e-\dot{e})^2]^{1/2}\{[1-(e-\dot{e})^2]^{1/2}-1\} = 1-[1-(e-\dot{e})^2]^{1/2}-(e-\dot{e})^2$,

in order to transform the denominator of the second term in the relation (81). In turn, we are able to perform a cancellation with the multiplier $\{-\{1-[1-(e-\dot{e})^2]^{1/2}\}\}$ in the nominator. Combining the two terms into one, we obtain:

(84) $$\textbf{\textit{solution } } \boldsymbol{a_1b_1} = \pi[1-(e-\dot{e})^2]^{-1/2} ln\{e^2\{4-6e^2+2e^4+6e\dot{e}-4e^3\dot{e}-2\dot{e}^2+2e^2\dot{e}^2+ \\ +(-4+4e^2-6e\dot{e}+2\dot{e}^2)(1-e^2)^{1/2}+(-4+4e^2-2e\dot{e})[1-(e-\dot{e})^2)]^{1/2}+ \\ +(4-2e^2+2e\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2)]^{1/2}\}\{2[1-(1-e^2)^{1/2}]e^2(e-\dot{e})^2\}^{-1}\} = \\ = \pi[1-(e-\dot{e})^2]^{-1/2} ln\{\{2-3e^2+e^4+3e\dot{e}-2e^3\dot{e}-\dot{e}^2+e^2\dot{e}^2+ \\ +(-2+2e^2-3e\dot{e}+\dot{e}^2)(1-e^2)^{1/2}+(-2+2e^2-e\dot{e})[1-(e-\dot{e})^2)]^{1/2}+ \\ +(2-e^2+e\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2)]^{1/2}\}\{[1-(1-e^2)^{1/2}](e-\dot{e})^2\}^{-1}\}.$$

It is interesting to check what will be the behaviour of the above solution under the transition $\dot{e}(u) \to 0$. We compute that:

(85) $$\lim_{\dot{e}(u)\to 0} \textbf{\textit{solution } } \boldsymbol{a_1b_1} = \pi(1-e^2)^{-1/2} ln\{[2-3e^2+e^4+(-2+2e^2)(1-e^2)^{1/2}+ \\ +(-2+2e^2)(1-e^2)^{1/2}+2-e^2-2e^2+e^4]\{e^2[1-(1-e^2)^{1/2}]\}^{-1}\} = \\ = \pi(1-e^2)^{-1/2} ln\{2[2-3e^2+e^4-2(1-e^2)(1-e^2)^{1/2}]\{e^2[1-(1-e^2)^{1/2}]\}^{-1}\} = \\ = \pi(1-e^2)^{-1/2} ln\{2(1-e^2)[2-e^2-2(1-e^2)^{1/2}]\{e^2[1-(1-e^2)^{1/2}]\}^{-1}\}.$$

We see that:

(86) $[1-(1-e^2)^{1/2}][1+(1-e^2)^{1/2}] = 1-(1-e^2) \equiv e^2$.

Consequently:

(87) $$[2-e^2-2(1-e^2)^{1/2}]\{e^2[1-(1-e^2)^{1/2}\}^{-1} = \\ = [2-e^2-2(1-e^2)^{1/2}][1+(1-e^2)^{1/2}]^{-1}[1-(1-e^2)^{1/2}]^{-2} = \\ = [2-e^2-2(1-e^2)^{1/2}][1+(1-e^2)^{1/2}]^{-1}[1-2(1-e^2)^{1/2}+1-e^2]^{-1} = \\ = 1/[1+(1-e^2)^{1/2}].$$

Substitution of (87) into (85) gives (see equality (42)):

(88) $$\lim_{\dot{e}(u)\to 0} \textbf{\textit{solution } } \boldsymbol{a_1b_1} = \pi(1-e^2)^{-1/2} ln\{2(1-e^2)/[1+(1-e^2)^{1/2}]\} = \textbf{\textit{solution 1}}.$$

Consequently:

(89) $$\lim_{\dot{e}(u)\to 0} (1/2)\mathbf{K_1}(e,\dot{e}) = (1/2)\mathbf{L_0}(e),$$

as we expected to be, in order to have an agreement between the definitions (3) and (4).

<u>Case II:</u> $\boldsymbol{a_1 = [-1+(1-e^2)^{1/2}]/e,\ b_2 = \{-1-[1-(e-\dot{e})^2]^{1/2}\}(e-\dot{e})^{-1}}$.

For this case $|a_1| < 1$ and $|b_2| > 1$. We evaluate that:

(90) $1 + (b_2)^2 = 1 + \{1 + 1 - (e - \dot{e})^2 + 2[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2} =$
$= 2\{1 + [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2},$

(91) $|1 - (b_2)^2| \equiv (b_2)^2 - 1 = \{1 + 1 - (e - \dot{e})^2 + 2[1 - (e - \dot{e})^2]^{1/2} - (e - \dot{e})^2\}(e - \dot{e})^{-2} =$
$= 2\{1 - (e - \dot{e})^2 + [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2} = 2\{\{[1 - (e - \dot{e})^2]^{1/2}\}^2 +$
$+ [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2} = 2[1 - (e - \dot{e})^2]^{1/2}\{1 + [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2}.$

Further we compute:

(92) $1 - (a_1/b_2) = 1 - [-1 + (1 - e^2)^{1/2}](e - \dot{e})\{e\{-1 - [1 - (e - \dot{e})^2]^{1/2}\}\}^{-1} =$
$= \{e + e[1 - (e - \dot{e})^2]^{1/2} - e + e(1 - e^2)^{1/2} + \dot{e} - \dot{e}(1 - e^2)^{1/2}\}\times$
$\times\{e\{1 + [1 - (e - \dot{e})^2]^{1/2}\}\}^{-1} =$
$= \{e(1 - e^2)^{1/2} + e[1 - (e - \dot{e})^2]^{1/2} + \dot{e} - \dot{e}(1 - e^2)^{1/2}\}\{e\{1 + [1 - (e - \dot{e})^2]^{1/2}\}\}^{-1}.$

(93) $(1/2)\mathbf{K_1}(e,\dot{e}) = \boldsymbol{solution\ a_1b_2} = -\pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(2/e^2)[1 - (1 - e^2)^{1/2}]\} +$
$+ 2\pi(e - \dot{e})^2\{1 + [1 - (e - \dot{e})^2]^{1/2}\}\{2(e - \dot{e})^2[1 - (e - \dot{e})^2]^{1/2}\{1 + [1 - (e - \dot{e})^2]^{1/2}\}\}^{-1}\times$
$\times ln\{\{e(1 - e^2)^{1/2} + e[1 - (e - \dot{e})^2]^{1/2} + \dot{e} - \dot{e}(1 - e^2)^{1/2}\}\{e\{1 + [1 - (e - \dot{e})^2]^{1/2}\}\}^{-1}\} =$
$= \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{e^2\{e^2 - e^4 + e^2 - e^2(e^2 + \dot{e}^2 - 2e\dot{e}) + \dot{e}^2 + \dot{e}^2 - e^2\dot{e}^2 +$
$+ 2e^2(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2} + 2e\dot{e}(1 - e^2)^{1/2} - 2e\dot{e}(1 - e^2) + 2e\dot{e}[1 - (e - \dot{e})^2]^{1/2} -$
$- 2e\dot{e}(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2} - 2\dot{e}^2(1 - e^2)^{1/2}\}\{2e^2[1 - (1 - e^2)^{1/2}]\times$
$\times\{1 + [1 - (e - \dot{e})^2]^{1/2}\}^2\}^{-1}\} =$
$= \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{\{e^2 - e^4 - e\dot{e} + 2e^3\dot{e} + \dot{e}^2 - e^2\dot{e}^2 + (e\dot{e} - \dot{e}^2)(1 - e^2)^{1/2} +$
$+ e\dot{e}[1 - (e - \dot{e})^2]^{1/2} + (e^2 - e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}\{[1 - (1 - e^2)^{1/2}]\times$
$\times\{1 + [1 - (e - \dot{e})^2]^{1/2}\}^2\}^{-1}\}.$

In order to simplify the argument of the logarithm, we evaluate its two multipliers:

(94) $\{[1 - (1 - e^2)^{1/2}]\{1 + [1 - (e - \dot{e})^2]^{1/2}\}^2\}^{-1} = \{1 - [1 - (e - \dot{e})^2]^{1/2}\}\times$
$\times\{[1 - (1 - e^2)^{1/2}]\{1 + [1 - (e - \dot{e})^2]^{1/2}\}\{1 + [1 - (e - \dot{e})^2]^{1/2}\}\times$
$\times\{1 - [1 - (e - \dot{e})^2]^{1/2}\}\}^{-1} =$
$= \{1 - [1 - (e - \dot{e})^2]^{1/2}\}\{[1 - (1 - e^2)^{1/2}]\{1 + [1 - (e - \dot{e})^2]^{1/2}\}[1 - 1 + (e - \dot{e})^2]\}^{-1} =$
$= \{1 - [1 - (e - \dot{e})^2]^{1/2}\}\{(e - \dot{e})^2[1 - (1 - e^2)^{1/2}]\{1 + [1 - (e - \dot{e})^2]^{1/2}\}\}^{-1},$

(95) $\{1 - [1 - (e - \dot{e})^2]^{1/2}\}\{e^2 - e^4 - e\dot{e} + 2e^3\dot{e} + \dot{e}^2 - e^2\dot{e}^2 + (e\dot{e} - \dot{e}^2)(1 - e^2)^{1/2} +$
$+ e\dot{e}[1 - (e - \dot{e})^2]^{1/2} + (e^2 - e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\} = (e - \dot{e})^2(1 - e^2 + e\dot{e}) -$
$- (e - \dot{e})^2(1 - e^2 + e\dot{e})(1 - e^2)^{1/2} - (e - \dot{e})^2(1 - e^2)[1 - (e - \dot{e})^2]^{1/2} +$
$+ (e - \dot{e})^2(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2} = (e - \dot{e})^2\{(1 - e^2 + e\dot{e})[1 - (1 - e^2)^{1/2}] +$
$+ (1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}[1 - (1 - e^2)^{1/2}]\} =$
$= (e - \dot{e})^2[1 - (1 - e^2)^{1/2}]\{1 - e^2 + e\dot{e} + (1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}.$

Substitution of the above results (94) and (95) into (93) gives:

(96) $(1/2)\mathbf{K_1}(e,\dot{e}) = \boldsymbol{solution\ a_1b_2} = \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(e - \dot{e})^2[1 - (1 - e^2)^{1/2}]\times$
$\times\{1 - e^2 + e\dot{e} + (1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}\{(e - \dot{e})^2[1 - (1 - e^2)^{1/2}]\times$
$\times\{1 + [1 - (e - \dot{e})^2]^{1/2}\}\}^{-1}\} =$
$= \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{\{1 - e^2 + e\dot{e} +$
$+ (1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}\{1 + [1 - (e - \dot{e})^2]^{1/2}\}^{-1}\}.$

It is easy to see that the transition $\dot{e}(u) \to 0$ gives the expected result:

(97) $\lim_{\dot{e}(u) \to 0} \boldsymbol{solution a_1b_2} = \lim_{\dot{e}(u) \to 0}[\pi[1 - (e - \dot{e})^2]^{-1/2} ln\{\{1 - e^2 + e\dot{e} +$

$+ (1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}\{1 + [1 - (e - \dot{e})^2]^{1/2}\}^{-1}\}] =$
$= \pi(1 - e^2)^{-1/2} ln\{2(1 - e^2)[1 + (1 - e^2)^{1/2}]^{-1}\} = (1/2)\mathbf{L_0}(e).$

There arises the natural question: whether the coincidence of the *solution* $\boldsymbol{a_1b_1}$ and the *solution* $\boldsymbol{a_1b_2}$ happens only in the limit $\dot{e}(u) \to 0$, or it is due to the equivalence of these solutions in general ? We shall show that the later situation is true. For this purpose, it is enough to check the equality of the arguments of the logarithmic functions. In fact, this means to verify that:

(98) $\{1 + [1 - (e - \dot{e})^2]^{1/2}\}\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 +$
$+ (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} + (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2)]^{1/2} +$
$+ (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2)]^{1/2}\} =$
$= e^2 - e^4 - 3e^2\dot{e} + (-e^2 + e^4 + 2e\dot{e} - 3e^3\dot{e} - \dot{e}^2 + 3e^2\dot{e}^2 - e\dot{e}^3)(1 - e^2)^{1/2} +$
$+ (-e^2 + e^4 + 2e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2)[1 - (e - \dot{e})^2]^{1/2} - 2e\dot{e} + 3e^3\dot{e} + e\dot{e}^3 + \dot{e}^2 +$
$+ (e^2 - 2e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2},$

(99) $[1 - (1 - e^2)^{1/2}](e - \dot{e})^2\{1 - e^2 + e\dot{e} + (1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\} = e^2 - e^4 - 2e\dot{e} +$
$+ 3e^3\dot{e} + \dot{e}^2 - 3e^2\dot{e}^2 + (-e^2 + e^4 + 2e\dot{e} - 3e^3\dot{e} - \dot{e}^2 + 3e^2\dot{e}^2 - e\dot{e}^3)(1 - e^2)^{1/2} +$
$+ (-e^2 + e^4 + 2e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2)[1 - (e - \dot{e})^2]^{1/2} +$
$+ (e^2 - 2e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}.$

The right-hand-sides of the above two equalities (98) and (99) are equal, which, in turn, after all, implies the equivalence of the ***solution*** $\boldsymbol{a_1b_1}$ (given by formula (84)) and ***solution*** $\boldsymbol{a_1b_2}$ (given by formula (96)).

<u>Case III:</u> $\boldsymbol{a_2 = [-1 - (1 - e^2)^{1/2}]/e,\ b_1 = \{-1 + [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-1}}$**.**

For this case $|a_2| > 1$ and $|b_1| < 1$. We have already computed that $1 + (a_2)^2 = (2/e^2)[1 + (1 - e^2)^{1/2}]$ (see equality (44)), which gives us the opportunity to write the expression (59) into the form:

(100) $(1/2)\mathbf{K_1}(e,\dot{e}) \equiv \int_0^\pi [ln(1 + e\cos\varphi)][1 + (e - \dot{e})\cos\varphi]^{-1} d\varphi =$ ***solution*** $\boldsymbol{a_2b_1} =$

$= -\pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(2/e^2)[1 + (1 - e^2)^{1/2}]\} - 2\pi\{1 - [1 - (e - \dot{e})^2]^{1/2}]\}\times$
$\times\{1 - [1 - (e - \dot{e})^2]^{1/2}] - (e - \dot{e})^2\}^{-1} ln|a_2 - b_1| =$
$= -\pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(2/e^2)[1 + (1 - e^2)^{1/2}]\} + \pi\{1 - [1 - (e - \dot{e})^2]^{1/2}]\}\times$
$\times\{[1 - (e - \dot{e})^2]^{1/2}]\{1 - [1 - (e - \dot{e})^2]^{1/2}]\}^{-1} ln(a_2 - b_1)^2 =$
$= \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(e^2/2)(a_2 - b_1)^2[1 + (1 - e^2)^{1/2}]^{-1}\}.$

We take into account that:

(101) $[1 + (1 - e^2)^{1/2}]^{-1} = [1 - (1 - e^2)^{1/2}]^2[1 - (1 - e^2)^{1/2}]^{-1}[1 - (1 - e^2)^{1/2}]^{-1}\times$
$\times[1 + (1 - e^2)^{1/2}]^{-1} = [2 - e^2 - 2(1 - e^2)^{1/2}]e^{-2}[1 - (1 - e^2)^{1/2}]^{-1}.$

(102) $(a_2 - b_1)^2 = \{[-1 - (1 - e^2)^{1/2}]/e - \{-1 + [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-1}\} =$
$= \{(e - \dot{e})(1 - e^2)^{1/2} + e - \dot{e} - e + e[1 - (e - \dot{e})^2]^{1/2}\}^2 e^{-2}(e - \dot{e})^{-2} =$
$= \{(e^2 + \dot{e}^2 - 2e\dot{e})(1 - e^2) + \dot{e}^2 + e^2 - e^2(e^2 + \dot{e}^2 - 2e\dot{e}) + (-2e\dot{e} + 2\dot{e}^2)(1 - e^2)^{1/2} +$
$+ (2e^2 - 2e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2} - 2e\dot{e}[1 - (e - \dot{e})^2]^{1/2}\}e^{-2}(e - \dot{e})^{-2} =$
$= 2\{e^2 - e^4 - e\dot{e} + 2e^3\dot{e} + \dot{e}^2 - e^2\dot{e}^2 - \dot{e}(e - \dot{e})(1 - e^2)^{1/2} +$

$$+ e(e - \dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2} - e\dot{e}[1 - (e - \dot{e})^2]^{1/2}\}e^{-2}(e - \dot{e})^{-2}.$$

Having in mind the above intermediate calculations (101) and (102), we are able to rewrite the expression (100) in the following way:

(103) $$(1/2)\,\mathbf{K_1}(e,\dot{e}) = \boldsymbol{solution\ a_2b_1} = \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{2e^2[2 - e^2 - 2(1 - e^2)^{1/2}\times \times\{e^2 - e^4 - e\dot{e} + 2e^3\dot{e} + \dot{e}^2 - e^2\dot{e}^2 - e(e - \dot{e})(1 - e^2)^{1/2} + + e(e - \dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2} - e\dot{e}[1 - (e - \dot{e})^2]^{1/2}\}\times \times\{2e^2[1 - (1 - e^2)^{1/2}e^2(e - \dot{e})^2\}^{-1}\} = = \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 + + (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} + (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} + + (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2}[1 - (1 - e^2)^{1/2}]^{-1}\} = = \boldsymbol{solution\ a_1b_1}.$$

To establish the equivalence of the *solution* $\boldsymbol{a_2b_1}$ with the *solution* $\boldsymbol{a_1b_1}$, we have used the result (84).

<u>Case IV:</u> $\boldsymbol{a_2 = [-1 - (1 - e^2)^{1/2}]/e,\ b_2 = \{-1 - [1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-1}}$**.**

For this case $|a_2| > 1$ and $|b_2| > 1$. Using the already computed expression for $1 + (a_2)^2$ (equality (44)), we have, according to formula (59), the following solution for the integral $\mathbf{K_1}(e,\dot{e})$:

(104) $$(1/2)\mathbf{K_1}(e,\dot{e}) \equiv \int_0^{\pi}[ln(1 + e cos\varphi)][1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi = \boldsymbol{solution\ a_2b_2} = = -\pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(2/e^2)[1 + (1 - e^2)^{1/2}]\} + 2\pi[1 - (e - \dot{e})^2]^{-1/2}]\times \times ln|a_2 - 1/b_2| = \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(e^2/2)(a_2 - 1/b_2)^2[1 + (1 - e^2)^{1/2}]^{-1}\} = = \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(e^2/2)[1 + 1 - e^2 - 2(1 - e^2)^{1/2}](a_2 - 1/b_2)^2\times \times[1 - (1 - e^2)^{1/2}]^{-1}[1 - (1 - e^2)^{1/2}]^{-1}[1 + (1 - e^2)^{1/2}]^{-1}\} = = \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{(e^2/2)[2 - e^2 - 2(1 - e^2)^{1/2}](a_2 - 1/b_2)^2[1 - (1 - e^2)^{1/2}]^{-1}\times \times(1 - 1 + e^2)^{-1}\}.$$

It remains to calculate the multiplier $(a_2 - 1/b_2)^2$, in order to finish the evaluation of the integral $\mathbf{K_1}(e,\dot{e})$ in this last **<u>Case IV</u>**.

(105) $$(a_2 - 1/b_2)^2 = \{[-1 - (1 - e^2)^{1/2}]/e - (e - \dot{e})\{-1 - [1 - (e - \dot{e})^2]^{1/2}\}^{-1}\}^2 = = \{[1 + (1 - e^2)^{1/2}]\{1 + [1 - (e - \dot{e})^2]^{1/2}\} - e(e - \dot{e})\}^2 e^{-2}\{1 + [1 - (e - \dot{e})^2]^{1/2}\}^{-2} = = \{1 - e^2 + e\dot{e} + (1 - e^2)^{1/2} + [1 - (e - \dot{e})^2]^{1/2} + (1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\times \times e^{-2}\{1 + [1 - (e - \dot{e})^2]^{1/2}\}^{-2}\} = 2\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 + + (2 - 2e^2 + 3e\dot{e} - \dot{e}^2)(1 - e^2)^{1/2} + (2 - 2e^2 + e\dot{e})[1 - (e - \dot{e})^2]^{1/2} + + (2 - 2e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}e^{-2}\{1 + [1 - (e - \dot{e})^2]^{1/2}\}^{-2}.$$

Substitution of the above equality into (104) leads to:

(106) $$(1/2)\mathbf{K_1}(e,\dot{e}) = \boldsymbol{solution\ a_2b_2} = \pi[1 - (e - \dot{e})^2]^{-1/2} ln\{2[2 - e^2 - 2(1 - e^2)^{1/2}\times \times\{1 - [1 - (e - \dot{e})^2]^{1/2}\}^2\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 + + (2 - 2e^2 + 3e\dot{e} - \dot{e}^2)(1 - e^2)^{1/2} + (2 - 2e^2 + e\dot{e})[1 - (e - \dot{e})^2]^{1/2} + + (2 - 2e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}\{2e^2[1 - (1 - e^2)^{1/2}]\times \times\{1 + [1 - (e - \dot{e})^2]^{1/2}\}^2\{1 - [1 - (e - \dot{e})^2]^{1/2}\}^2\}^{-1}\},$$

where we have multiplied *both* the nominator and the denominator of the argument of the logarithm by the *same* multiplier $\{1-[1-(e-\dot{e})^2]^{1/2}\}^2$.

We also have the equality:

(107) $\{1+[1-(e-\dot{e})^2]^{1/2}\}^2\{1-[1-(e-\dot{e})^2]^{1/2}\}^2 = [1-1+(e-\dot{e})^2]^2 \equiv (e-\dot{e})^4$.

After some tedious algebra, we arrive at the final expression for $\mathbf{K_1}(e,\dot{e})$:

(108) $(1/2)\mathbf{K_1}(e,\dot{e}) = \boldsymbol{solution\ a_2b_2} = \pi[1-(e-\dot{e})^2]^{-1/2} ln\{e^2(e-\dot{e})^2\{2-3e^2+e^4+$
$+3e\dot{e}-2e^3\dot{e}-\dot{e}^2+e^2\dot{e}^2+(-2+2e^2-3e\dot{e}+\dot{e}^2)(1-e^2)^{1/2}+$
$+(-2+2e^2-e\dot{e})[1-(e-\dot{e})^2]^{1/2}+(2-e^2+e\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{1/2}\}\times$
$\times\{e^2(e-\dot{e})^4[1-(1-e^2)^{1/2}]\}^{-1}\} = \boldsymbol{solution\ a_1b_1}$.

In this way, we obtain that for all the possible cases, prescribed by the formula (59) for the different combinations $\{a_i, b_j\}$, (i, j = 1, 2) of the roots a_1, a_2, b_1 and b_2, the solutions for the integral $\mathbf{K_1}(e,\dot{e})$ are equivalent. Of course, it is reasonable to check whether these evaluations remain valid under these values of the variables $e(u)$, $\dot{e}(u)$ and $e(u)-\dot{e}(u)$, when we strike with nullification of some of the denominators in the intermediate calculations. For example, if $e(u) = 0$, we have that:

(109) $\mathbf{K_1}(0,\dot{e}) \equiv \int_0^{2\pi}[ln(1)](1+\dot{e}cos\varphi)^{-1}\,d\varphi = 0.$

At the same time, from formula (84) (describing the *solution* $\boldsymbol{a_1b_1}$), we may evaluate the factor in the argument of the logarithmic function, which is associated with the "peculiar" behavior under the limit transition $e(u) \to 0$. Omitting the multiplier $1/(e-\dot{e})^2$, which tends to $1/\dot{e}^2$, when $e(u) \to 0$, we have to compute the following limit:

(110) $\lim_{e(u)\to 0}\{\{2-3e^2+e^4+3e\dot{e}-2e^3\dot{e}-\dot{e}^2+e^2\dot{e}^2+(-2+2e^2-3e\dot{e}+\dot{e}^2)(1-e^2)^{1/2}+$
$+(-2+2e^2-e\dot{e})[1-(e-\dot{e})^2]^{1/2}+(2-e^2+e\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{1/2}\}\times$
$\times\{[1-(1-e^2)^{1/2}]\}^{-1}\}$.

Because for the denominator we have:

(111) $\lim_{e(u)\to 0}\{\partial/\partial e[1-(1-e^2)^{1/2}]\} = \lim_{e(u)\to 0}[e(1-e^2)^{-1/2}] = 0,$

(112) $\lim_{e(u)\to 0}\{\partial/\partial e[e(1-e^2)^{-1/2}]\} = \lim_{e(u)\to 0}[(1-e^2)^{-1/2}+e^2(1-e^2)^{-3/2}] = 1.$

This means that if we want to evaluate the expression (110) by means of the L'Hospital's rule, we need to apply it two successive times. It is easily verified that the conditions for such an approach are fulfilled. In fact, we have to calculate the second derivative of the nominator in the equality (110), and than to take the limit $e(u) \to 0$.

$$(113)\quad \lim_{e(u)\to 0}\{\partial^2/\partial e^2\{2-3e^2+e^4+3e\dot{e}-2e^3\dot{e}-\dot{e}^2+e^2\dot{e}^2+$$
$$+(-2+2e^2-3e\dot{e}+\dot{e}^2)(1-e^2)^{1/2}+(-2+2e^2-e\dot{e})[1-(e-\dot{e})^2]^{1/2}+$$
$$+(2-e^2+e\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{1/2}\}\}=\lim_{e(u)\to 0}\{\partial/\partial e\{-6e+4e^3+3\dot{e}-6e^2\dot{e}+$$
$$+2e\dot{e}^2+(4e-3\dot{e})(1-e^2)^{1/2}+(2e-2e^3+3e^2\dot{e}-e\dot{e}^2)(1-e^2)^{-1/2}+$$
$$+(4e-\dot{e})[1-(e-\dot{e})^2]^{1/2}+(2e-2e^3-2\dot{e}+3e^2\dot{e}-e\dot{e}^2)[1-(e-\dot{e})^2]^{-1/2}+$$
$$+(-2e+\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{1/2}+(-2e+e^3-e^2\dot{e})(1-e^2)^{-1/2}\times$$
$$\times[1-(e-\dot{e})^2]^{1/2}+(-2e+e^3+2\dot{e}-2e^2\dot{e}+e\dot{e}^2)(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{-1/2}\}=$$
$$=\lim_{e(u)\to 0}\{-6+12e^2-12e\dot{e}+2\dot{e}^2+4(1-e^2)^{1/2}-(4e-3\dot{e})e(1-e^2)^{-1/2}+$$
$$+(2-6e^2+6e\dot{e}-\dot{e}^2)(1-e^2)^{-1/2}+(2e-2e^3+3e^2\dot{e}-e\dot{e}^2)e(1-e^2)^{-3/2}-$$
$$-(4e-\dot{e})(e-\dot{e})[1-(e-\dot{e})^2]^{-1/2}+(2-6e^2+6e\dot{e}-\dot{e}^2)[1-(e-\dot{e})^2]^{-1/2}+$$
$$+(2e-2e^3-2\dot{e}+3e^2\dot{e}-e\dot{e}^2)(e-\dot{e})[1-(e-\dot{e})^2]^{-3/2}-$$
$$-2(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{1/2}+(2e-\dot{e})e(1-e^2)^{-1/2}[1-(e-\dot{e})^2]^{1/2}+$$
$$+(2e-\dot{e})(e-\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{-1/2}+$$
$$+(-2+3e^2-2e\dot{e})(1-e^2)^{-1/2}[1-(e-\dot{e})^2]^{1/2}+(-2e+e^3-e^2\dot{e})e(1-e^2)^{-3/2}\times$$
$$\times[1-(e-\dot{e})^2]^{1/2}+(-2e+e^3-e^2\dot{e})(e-\dot{e})(1-e^2)^{-1/2}[1-(e-\dot{e})^2]^{-1/2}+$$
$$+(-2+3e^2-4e\dot{e}+\dot{e}^2)(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{-1/2}-$$
$$-(-2e+e^3+2\dot{e}-2e^2\dot{e}+e\dot{e}^2)e(1-e^2)^{-1/2}+$$
$$+(-2e+e^3+2\dot{e}-2e^2\dot{e}+e\dot{e}^2)(e-\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{-3/2}\}=$$
$$=-6+2\dot{e}^2+4+2-\dot{e}^2+4(1-\dot{e}^2)^{1/2}-\dot{e}^2(1-\dot{e}^2)^{-1/2}+(2-\dot{e}^2)(1-\dot{e}^2)^{-1/2}-$$
$$-2(1-\dot{e}^2)^{1/2}+\dot{e}^2(1-\dot{e}^2)^{-1/2}-2(1-\dot{e}^2)^{1/2}+(-2+\dot{e}^2)(1-\dot{e}^2)^{-1/2}-$$
$$-\quad 2\dot{e}^2(1-\dot{e}^2)^{-1}(1-\dot{e}^2)^{-1/2}=\dot{e}^2.$$

Consequently (using two times the L'Hospital's rule), we have:

$$(114)\quad \{\lim_{e(u)\to 0}(e-\dot{e})^{-2}\}\lim_{e(u)\to 0}\{[1-(1-e^2)^{1/2}]^{-1}\{2-3e^2+e^4+3e\dot{e}-2e^3\dot{e}-\dot{e}^2+e^2\dot{e}^2+$$
$$+(-2+2e^2-3e\dot{e}+\dot{e}^2)(1-e^2)^{1/2}+(-2+2e^2-e\dot{e})[1-(e-\dot{e})^2]^{1/2}+$$
$$+(2-e^2+e\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{1/2}\}\}=(1/\dot{e}^2)\dot{e}^2=1.$$

It seems out that the argument of the logarithm in the *solution* $\boldsymbol{a_1b_1}$ approaches unity, when $e(u)\to 0$, and, correspondingly, the value of the logarithm approaches zero. This is in agreement with the direct computation of the integral $\mathbf{K_1}(e,\dot{e})$, when $e(u)=0$ (see equality (109)).

As regards to the situation when $e(u)-\dot{e}(u)=0$ (this possibility is excluded *a priori* by hypothesis during the calculation of the expression (84)), a *direct* computation of the integral $\mathbf{K_1}(e,\dot{e}=e)$ gives:

$$(115)\quad (1/2)\,\mathbf{K_1}(e,\dot{e}=e)\equiv\int_0^{\pi} ln(1+e\cos\varphi)\,d\varphi=\pi\, ln\{[1+(1-e^2)^{1/2}]/2\}.$$

Here we have used formula 865.44 from Dwight [6], setting in it $a =$ $= 1$ and $b = e(u)$, and taking into account that for the all parts of the accretion disc $e(u)$ is less than unity (by absolute value).The transition $e(u) -$ $-\dot{e}(u)\to 0$ may be attained in two ways: (*i*) by fixing $\dot{e}(u)$ and letting $e(u)$ to approach $\dot{e}(u)$, and (*ii*) by fixing $e(u)$ and letting $\dot{e}(u)$ to approach $e(u)$. If we apply these two methods to the expression (84), drawing the correspondingly times the L'Hospital's rule for revealing of indeterminacies of the type 0/0, we shall obtain a result which is identical to the relation (115). This means that the formula (84) can be useful also in the case when $e(u) - \dot{e}(u) = 0$, despite it was derived under the rejection of the later equality. It is important only to remember that in this "peculiar" case it is necessary to perform the limit transition $e(u) - \dot{e}(u) \to 0$. This transition gives also a continuous passage of the integral $\mathbf{K_1}(e,\dot{e})$ through the point $e(u) - \dot{e}(u) = 0$. We shall not write here the tedious computations, which prove the above statements. We restrict us only to mention that they are valid, in order to underline that the formula (84) (respectively, *solution* $\boldsymbol{a_1b_1}$ = *solution* $\boldsymbol{a_1b_2}$ = *solution* $\boldsymbol{a_2b_1}$ = *solution* $\boldsymbol{a_2b_2}$) is not limited by any restrictions, imposed by the values of the eccentricity $e(u)$ and its derivative $\dot{e}(u) \equiv de(u)/du$. Of course, the quantities $e(u)$ and $\dot{e}(u)$ oneself must obey the three inequalities $|e(u)| < 1$, $|\dot{e}(u)| < 1$ and $|e(u) - \dot{e}(u)| < 1$. They are induced by the properties of the considered elliptical accretion disc model [1], as mentioned earlier. To end this chapter, we write into the final form the analytical expression for the integral $\mathbf{K_1}(e,\dot{e})$. Taking into account that the transition of the integration over the azimuthal angle φ from the interval $[0, \pi]$ to the interval $[0, 2\pi]$ simply leads to a doubling of the result, we are able to give the following analytical formula:

$$(116) \quad \mathbf{K_1}(e,\dot{e}) \equiv \int_0^{2\pi} [ln(1 + e\cos\varphi)][1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi =$$

$$= 2\pi[1 - (e - \dot{e})^2]^{-1/2} ln\{\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 +$$
$$+ (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} + (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} +$$
$$+ (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2}[1 - (1 - e^2)^{1/2}]^{-1}\}.$$

3. Conclusions

In the present paper we have moved one step more towards the revealing of the mathematical characteristics of the dynamical equation. The later determines the *spatial* structure of the *stationary* elliptical accretion discs, according to the model of Lyubarskij et al. [1]. More concretely, it is shown, that we are able to perform analytical evaluations of two kinds of integrals, which are functions of the eccentricity $e(u)$ and its derivative $\dot{e}(u) \equiv de(u)/du$. Namely, these are $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, ..., 3) and $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, ..., 5), defined by the equalities (3) and (4), respectively. It is possible to calculate analytical expressions for the integrals $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 1, 2, 3) through recurrence relations, under the condition that both the lower order (in the sense of the indices **i** and **j**) integrals $\mathbf{L_i}(e,\dot{e})$ and $\mathbf{K_j}(e,\dot{e})$ are already known. About the computation of the integrals $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, ..., 5) the situation is slightly different. There is not need to know the expressions for $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, ..., 3), but only these for the other integrals $\mathbf{K_m}(e,\dot{e})$, ($\mathbf{m}$ = 1, ..., $\mathbf{j}$ – 1). In preparation to solve the so mentioned two kinds of integrals, we have computed the "initial" integrals $\mathbf{L_0}(e)$ and $\mathbf{K_1}(e,\dot{e})$, which are recognized to serve as starting points for the established recurrence relations. The complete set of analytical solutions for the integrals $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 1, 2, 3) and $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 2, ..., 5) will be expressed in a forthcoming paper [7]. Until now, we have traced out the way to reach the determination of these analytical formulas. As follows from the evaluations of $\mathbf{L_0}(e)$ and $\mathbf{K_1}(e,\dot{e})$, we strike with somewhat tedious calculations of these two integrals. But nevertheless, they lead to the pleasurable conclusion that all possible combinations of the permitted values of the parameters give identical solutions for the integrals $\mathbf{L_0}(e)$ and $\mathbf{K_1}(e,\dot{e})$. This property, i.e., the uniqueness of the solutions, obviously facilitates our task to find the analytical solutions of the integrals $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, ..., 3) and $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, ..., 5).

References

1. L y u b a r s k i j, Y u. E., K. A. P o s t n o v, M. E. P r o k h o r o v. Eccentric accretion discs., Monthly Not. Royal Astron. Soc., 266, 1994, № 2, pp. 583–596.
2. D i m i t r o v, D. V. Thin viscous elliptical accretion discs with orbits sharing a common longitude of periastron. V. Linear relations between azimuthal-angle averaged factors in the dynamical equation., Aerospace Research in Bulgaria, 24, 2012.
3. D i m i t r o v, D. V. Thin viscous elliptical accretion discs with orbits sharing a common longitude of periastron. VI. Simplification of the dynamical equation., Aerospace Research in Bulgaria, 24, 2012.
4. D i m i t r o v, D. V. Analytical computation of two integrals, appearing in the theory of elliptical accretion discs. I. Solving of the auxiliary integrals, emerging during their derivations., Aerospace Research in Bulgaria, 25, 2013.
5. P r u d n i k o v, A. P., Y u. A. B r i c h k o v, O. I. M a r i c h e v. Integrals and series. Elementary functions, Moscow, "Nauka" publishing house, 1981 (in Russian).
6. D w i g h t, H. B. Tables of integrals and other mathematical data., Fourth edition, New York, MacMillan company, 1961.
7. D i m i t r o v, D. V. Analytical computation of two integrals, appearing in the theory of elliptical accretion discs. I. Solving of the full set of auxiliary integrals, containing logarithmic functions into their integrands, Aerospace Research in Bulgaria, 26, 2014, (to appear).

АНАЛИТИЧНО ПРЕСМЯТАНЕ НА ДВА ИНТЕГРАЛА, ВЪЗНИКВАЩИ В ТЕОРИЯТА НА ЕЛИПТИЧНИТЕ АКРЕЦИОННИ ДИСКОВЕ. II. РЕШАВАНЕ НА НЯКОИ СПОМАГАТЕЛНИТЕ ИНТЕГРАЛИ, СЪДЪРЖАЩИ ЛОГАРИТМИЧНИ ФУНКЦИИ В СВОИТЕ ИНТЕГРАНДИ

Д. Димитров

Резюме

Тази статия е част от изследванията, третиращи математическата структура на *стационарните* елиптични акреционни дискове в модела на Любарски и др. [1], т.е., дискове при които всички апсидни линии на орбитите на частиците лежат върху една и съща права линия. Главната отличителна черта на възприетия подход е да се намерят линейни зависимости между интегралите, влизащи в това уравнение. Те ще ни дадат възможност да елиминираме тези сложни (и изобщо казано, неизвестни в аналитичен вид) функции на ексцентрицитета $\boldsymbol{e(u)}$ и неговата производна $\boldsymbol{\dot{e}(u) \equiv de(u)/du}$ на орбитите. Тук $\boldsymbol{u \equiv ln(p)}$, където $\boldsymbol{p}$ е фокалният параметър на орбитата на съответната частица от акреционния диск. В течение на процеса на реализиране на тази програма, ние се сблъскваме с необходимостта да намерим аналитични оценки за два вида интеграли:

$L_i(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-i}\, d\varphi$, ($i$ = 0,…, 3), and $K_j(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + ecos\varphi)]\times$ $\times[1 +(e - \dot{e})cos\varphi]^{-j}\, d\varphi$, ($j$ = 1, …, 5). В настоящето изследване, ние намираме рекурентни съотношения, даващи ни възможност да изчислим интегралите $\mathbf{L_i}(\boldsymbol{e,\dot{e}})$, (**i** = 1, .., 3) при условие че интегралите $\mathbf{L_{i-1}}(\boldsymbol{e,\dot{e}})$ и $\mathbf{K_i}(\boldsymbol{e,\dot{e}})$ са вече известни. Обратно, изчисленията на интегралите $\mathbf{K_j}(\boldsymbol{e,\dot{e}})$, (**j** = 1, ...,5), чрез рекурентни зависимости, *не изискват* знанието на аналитичните решения на интегралите $\mathbf{L_i}(\boldsymbol{e,\dot{e}})$, (**i** = 0, .., 3). С оглед на факта, че интегралите $\mathbf{L_0}(e)$ (той не зависи от $\boldsymbol{\dot{e}(u)}$) и $\mathbf{K_1}(e,\dot{e})$ служат като "отправни точки" в съответните рекурентни съотношения, ние сме намерили аналитични изрази за тях. Решаването на пълната система от аналитични оценки за $\mathbf{L_i}(\boldsymbol{e,\dot{e}})$, (**i** = 1, ..., 3), и $\mathbf{K_j}(\boldsymbol{e,\dot{e}})$, (**j** = 2, ...,5), ще бъде дадено другаде [7].