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# ANALYTICAL COMPUTATION OF TWO INTEGRALS, APPEARING IN THE THEORY OF ELLIPTICAL ACCRETION DISCS. I. SOLVING OF THE AUXILIARY INTEGRALS, EMERGING DURING THEIR DERIVATIONS

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***Abstract***

*The present work is a part of an extended analytical investigation of the dynamical equation, determining the spatial structure of the* **stationary** *elliptical accretion discs, according to the model of Lyubarskij et al. [1]. In the mathematical description of the problem are used as parameters the eccentricity* $e(u)$ *of the particle orbits, and its derivative* $\dot{e}(u) \equiv de(u)/du$ *,where* $u \equiv \ln(p)$*, and* $p$ *is the focal parameter of the considered orbit. During the process of simplification of that equation, there arises the necessity of analytical evaluations of integrals of the following types:* $A_i(e, \dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-i} d\varphi$, ($i$ = 1,…,5), $J_k(e, \dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-k} d\varphi$ *and* $H_k(e, \dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-k} \times [1 + (e - \dot{e})\cos\varphi]^{-1} d\varphi$, ($k$ = 1,…,4). *In these formulas* $\varphi$ *is the azimuthal angle, over which the averaging is taken. The approach in solving of the task is, in fact, recursive. At first, we evaluate the integrals with the smallest* $i$ *and* $k$ *(i.e.,* $i$ *and* $k$ *equal to unity). After then, we go to the next steps, gradually increasing the integer powers* $i$ *or* $k$*, until achieving the designated values 5 or 4, correspondingly. A special attention is devoted to these values of* $e(u)$ *and* $\dot{e}(u)$ *(and their difference* $e(u) - \dot{e}(u)$ *), which, eventually, may cause divergences in the intermediate or the final expressions. It is shown that although such troubles arise, they can be overcome by means of a* **direct** *substitution of the "peculiar" values of* $e(u)$ *and/or* $\dot{e}(u)$ *into the integrals, and after then performing the calculations. Even if the denominators in the final results appear factors equal to zero (due to the nullifications of* $e(u)$*,* $\dot{e}(u)$ *or* $e(u) - \dot{e}(u)$ *), the expressions are not divergent , as we have proved, using the L'Hospital's rule for resolving of indeterminacies of the type 0/0. All the analytical estimations of the above written integrals are performed under the restrictions*

*$|e(u)| < 1$, $|\dot{e}(u)| < 1$ and $|e(u) - \dot{e}(u)| < 1$. They are imposed by the physical reasons, in view of the application of these solutions into the adopted theory of the elliptical accretion discs.*

## 1. Introduction

We have considered some simplifications of the dynamical equation, governing the structure of the elliptical accretion discs in the model of Lyubarskij et al. [1]. The results are already published in a series of papers ([2], [3] and [4]; see also the references therein). In the course of this work, we have introduced seven integrals, which are functions of the eccentricities $e(u)$ of the particle orbits in the accretion disc, their derivatives $\dot{e}(u) \equiv de(u)/du$ and the power $n$ into the viscosity law $\eta = \beta\, \Sigma^{n}$. Further we explain the use of the introduced notations. Here $u$ is defined to be the logarithm of the focal parameter $p$ of the corresponding ellipse, representing the considered particle orbit: $u \equiv ln(p)$. We remind that in the considered model of Lyubarskij et al. [1], all elliptical trajectories in the accretion flow are such, that the major axes of the ellipses lie on the same line (assumed to be the abscissa on which lie the periastrons and apoastrons of the all trajectories). This simplification (introduced "by hands") allows to derive a dynamical equation for the particles of the disc, which is a second order ordinary differential equation [1]. Such a situation is more favorable, if we try to apply an analytical approach for solving of this problem. The picture of the dynamics of the elliptical accretion discs becomes much more complicated in the opposite (more general) case, when the ellipses of the orbits have apse lines, which are not necessarily in line with each other. Then the dynamics of the disc is described by partial differential equations, as it has been shown in the investigation of Ogilvie [5]. Our working out of the model of Lyubarskij et al. [1] is stimulated in the first place namely by the above mentioned simplifying circumstance, allowing more favorable possibilities to solve the problem by purely analytical methods. Though the considered case may have less usefulness with respect to the really observed discs. That is to say, elliptical discs with orbits sharing a common longitude of the periastron are rare situations among the objects of the kind eccentric accretion discs. It is worth to note that while the orbital eccentricity $e(u)$ and its derivative $\dot{e}(u) \equiv de(u)/du$ are functions of the focal parameter $p$ ($u \equiv ln(p)$), the power $n$ does not depend on $u$. This means that $n$ is a fixed constant through the whole disc, while the elongation of the particle orbits may vary for the different parts of the disc. In particular, for the *outer* parts the periastron of the elliptical orbits may have positive (negative) values of the abscissa, but at the same time in the *inner* parts of the accretion disc, such values may take negative (positive) meanings, respectively. As it has been written above, the accepted in [1] viscosity law is $\eta = \beta\,\Sigma^{n}$ (where $\beta$ is a constant). The viscosity parameter $\eta$ will depend on the spatial coordinates $r$ and $\varphi$ (where $r$ is the length of the radius-vector, measured from the center of the compact object, accreting the matter; $\varphi$ is the azimuthal angle) only through the surface density of the disc $\Sigma = \Sigma(r, \varphi)$.

During the process of simplification of the dynamical equation of the elliptical accretion discs (derived by Lyubarskij et al. [1], we have introduced seven auxiliary integrals, which appear because of the azimuthal-angle averaging of the task. These integrals are functions of $e(u)$, $\dot{e}(u)$ and $n$, and are defined in the following manner ([2], [3] and [4]):

$$(1)\qquad \mathbf{I_{0-}}(e,\dot{e},n) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{\,n-3}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi\ ,$$

$$(2)\qquad \mathbf{I_{0+}}(e,\dot{e},n) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{\,n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+2)}\, d\varphi\ ,$$

$$(3)\qquad \mathbf{I_{j}}(e,\dot{e},n) \equiv \int_0^{2\pi}(\cos\varphi)^{\mathrm{j}}(1 + e\cos\varphi)^{\,n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi\ ;\ \ \mathbf{j} = 0, 1, 2, 3, 4.$$

We remind here, that we consider *a particular* case of the elliptical accretion discs, developed by Lyubarskij et al. [1]: namely, stationary flows. Moreover, there are some additional limitations, imposed for every elliptical orbit in the disc, on the eccentricity $e(u)$ and its derivative $\dot{e}(u) \equiv de(u)/du$. These are the inequalities: $|e(u)| < 1$, $|\dot{e}(u)| < 1$ and $|e(u) - \dot{e}(u)| < 1$, valid for every value $u \equiv ln(p)$ in the disc. Mathematically viewed, these conditions ensure that the integrals (1) – (3) are well behaving, because the denominators are always strongly positive and, correspondingly, do not cause singularities. From a physical point of view, possible nullifications of the denominators in the definitions (1) – (3) might be connected with the emerging of shock waves in the disc, leading, in own turn, to spiral density waves [1]. Such phenomena *a priori* are not considered by this model. The discussed circumstance is clearly reflected in the expressions for the denominators of the metric tensor and the related quantities (see the Appendices in paper [1]).

In the above cited earlier investigations [2], [3] and [4], we have interested in the establishing of the linear relations between the integrals (1) – (3), in order to eliminate them from the dynamical equation of the accretion flow. This is in a correspondence with our approach to simplify analytically the equation and, eventually, to reveal its mathematical structure and physical implications by purely analytical manners. And only after that to apply, if it is unavoidable, the numerical computations. Leaving aside the integral $\mathbf{I_3}(e,\dot{e},n)$, we have shown that four of the other integrals (1) – (3) may be expressed through linear combinations of the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$. So that, to proceed further, we need to investigate whether the last two integrals are linearly independent functions with respect to the variables $e(u)$ and $\dot{e}(u)$, or not. We remark here that the power $n$ in the viscosity law $\eta = \beta\ \Sigma^{\ n}$ is a fixed quantity throughout the entire elliptical accretion disc. When we state that $n$ is a parameter, entering as an independent variable in the list of arguments of the integrals (1) – (3), etc., we subtend that we, in fact, consider a family of an infinite number of discs. Every with own fixed value of the power $n$. Saying that $n$ varies, we bear in mind that such a variation of $n$ is not over the spatial coordinates in the disc, but from one model to other model (with different $n$); i.e., $\boldsymbol{n}$ does not depend on $\boldsymbol{u \equiv ln(p)}$. This situation, of course, simplifies the differentiation with respect to $e(u)$ or $\dot{e}(u)$ of variety kinds of expressions, like $(1 + ecos\varphi)^n$, $[1 + (e - \dot{e})cos\varphi]^n$, etc. But there are some cases, when we need of the derivatives with respect to $n$. Then, according to the well known differentiating rule from the analysis $d(a^x)/dx = a^x\ ln(a)$ (where $a$ does not depend on $x$), as we shall see later, in the integrands of the considered integrals will appear factors of the type $ln(1 + ecos\varphi)$ and $ln[1 + (e - \dot{e})cos\varphi]$. This complicates the analytical computation of the integrals, because we did not successfully find any expressions about them in the accessible for us mathematical reference books, manuals, guides and handbooks. The reason for differentiating with respect to the power $n$ is the following. During the process of verification of linear dependence/independence of the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$, there appear terms containing into their denominators factors like $(n - 1)$, $(n - 2)$, etc., which implies suspicions of divergences, if we try to use the final results for some integer values of $n$. Of course, we are able to perform the evaluation of the considered expressions in a separate manner for these “peculiar” integer values of $n$ and obtain nonsingular results for this special cases. Such a possibility is guaranteed by the form of the initial expressions (namely, the integrals of the type (1) – (3) and the other integrals, originating from them), which we try to evaluate analytically. They are obviously not singular for these “problematic” integer values of $n$. But from

physical reasons, there is not motivation to assume the existence of such "special" selection of some integer $n$, and we expect that the pointed out property to be reflected into the mathematical formulas. More strictly speaking, we suspect that the divergences, appearing because of the nullification of the denominators for some integer $n$, may be overcome by means of the L'Hospital's rule for resolving of indeterminacies of the type 0/0. Such an additional checking of the results for the above mentioned "problematic" integer values of the power $n$ has two reasons: ($i$) the transition through these integer values of $n$ is continuos. That is to say, the direct computation of the analytically evaluated integrals gives the same results as in the case, when the limit transition to the "problematic" integer $n$ is used into the "singular" formulas. If the L'Hospital's rule may be applied, of course! There are two L'Hospital's rules: one helps us to evaluate indeterminacies of the type 0/0 and the other – for the type ∞/∞. In the our further exposition we shall use only the first theorem of L'Hospital. For this reason, let us formulate (in order to make things clear) the first variant of these rules. The proof of these statements can be found in many textbooks on differential calculus, and we shall not cite them in our references. Because the variables, which describe the accretion disc model, are real numbers, the formulation of the first L'Hospital's rule will be restricted to this case. Let we have a point $x_0$ (in our application, this may be a concrete value of $e(u)$, $\dot{e}(u)$, $e(u) - \dot{e}(u)$ or $n$). Let be fulfilled the following conditions: ($i$) functions $f(x)$ and $g(x)$ are defined and continuous in some interval around $x_0$; ($ii$) both these functions approach zero, when $x$ approaches $x_0$:
$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$; ($iii$) the derivatives $f'(x) \equiv df(x)/dx$ and $g'(x) \equiv dg(x)/dx$ in that interval (except, may be, at the point $x_0$) exist; ($iv$) these derivatives do not simultaneously vanish for $x \neq x_0$; ($v$) there also exists the limit
$\lim_{x \to x_0} [f'(x)/g'(x)]$.
Then, under these circumstances, the first L'Hospital's rule states that
$\lim_{x \to x_0} [f(x)/g(x)] = \lim_{x \to x_0} [f'(x)/g'(x)]$. In what follows, when arise the need of application of the L'Hospital's rule, the points ($i$) – ($v$) must be checked for their validity. If some of them are not obvious, we shall give a detailed proof of the correctness of these conditions. It may occur, that the rule has to

be applied several times successively, in order to be achieved the reasonable final result.

The establishing of the linear dependence/independence of the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$ follows the standard way – computing the Wronski determinant and evaluation of the domains in the space of variables, where it is equal (or not equal) to zero. In the course of this procedure, which we intend to perform in a purely analytical manner, without using numerical methods, we arrive to the problem of the analytical solving of two integrals. Like the definitions (1) – (3), they are also functions of $e(u)$, $\dot{e}(u)$ and the power $n$:

$$(4) \qquad \mathbf{I_{0,-4,+1}}(e,\dot{e},n) \equiv \int_0^{2\pi} (1 + e\cos\varphi)^{n-4} [1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi \,,$$

$$(5) \qquad \mathbf{I_{0,-2,+3}}(e,\dot{e},n) \equiv \int_0^{2\pi} (1 + e\cos\varphi)^{n-2} [1 + (e - \dot{e})\cos\varphi]^{-(n+3)}\, d\varphi \,.$$

The appearance of such expressions is a consequence of the differentiation of $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$, in order to write the Wronski determinant. In turn, the computation of the integrals (4) and (5) requires a preliminary analytical evaluation of some auxiliary integrals, also functions of $e(u)$, $\dot{e}(u)$ and $n$. We divide them into two groups, whether their integrands include (or not include) as factors the logarithms
$ln(1 + e\cos\varphi)$ and $ln[1 + (e - \dot{e})\cos\varphi]$.

## 2. Analytical computation of the auxiliary integrals, which do not contain logarithmic functions

### 2.1. Evaluation of integrals of the type $\mathbf{A_i}(e,\dot{e}) \equiv \int_0^{2\pi} [1 + (e - \dot{e})\cos\varphi]^{-i}\, d\varphi$

In the present subsection we calculate integrals with integrands which are *negative integer* powers of the expression $[1 + (e - \dot{e})\cos\varphi]$. As already mentioned above, we investigate the model of elliptical accretion discs of Lyubarskij et al. [1] under three restrictions, imposed *a priori* on the eccentricity $e = e(u)$, its derivative $\dot{e}(u) \equiv de(u)/du$ and the difference $e(u) - \dot{e}(u)$.They must be fulfilled for all parts of the accretion flow, i.e., for all $u \equiv ln(p)$. Particularly, $|\, e(u) - \dot{e}(u)| < 1$, which ensure that $[1 + (e - \dot{e})\cos\varphi]$ never vanishes for all values of the azimuthal angle φ $(0 \le \varphi \le 2\pi)$. With this remark, we are able to evaluate, without any complications, the integrals $\mathbf{A_i}(e,\dot{e})$, defined through the relation:

(6) $\quad \mathbf{A_i}(e,\dot{e}) \equiv \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-\,\mathrm{i}}\, d\varphi\,,$ $\quad$ **i** – non-negative integer.

Actually, we shall need of analytical expressions for $\mathbf{A_i}(e,\dot{e})$, when ***i*** = 1, 2, 3, 4 and 5. Note that these functions do not depend on the power *n*! According to formulas **858.525** and **858.535** from the tables of Dwight [6], we are able immediately to give the analytical expressions for $\mathbf{A_1}(e,\dot{e})$ and $\mathbf{A_2}(e,\dot{e})$, respectively:

(7) $$\mathbf{A_1}(e,\dot{e}) \equiv \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi = 2\pi\,[1 - (e - \dot{e})^2]^{-1/2}\,,$$

(8) $$\mathbf{A_2}(e,\dot{e}) \equiv \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-2}\, d\varphi = 2\pi\,[1 - (e - \dot{e})^2]^{-3/2}\,.$$

Further we observe that for a *fixed* value $n = 3$, the integral $\mathbf{I_{0\text{-}}}(e,\dot{e},n=$ $= 3)$ coincides with the function $\mathbf{A_4}(e,\dot{e})$ (see the definition (1) for $\mathbf{I_{0\text{-}}}(e,\dot{e},n)$):

(9) $$\mathbf{A_4}(e,\dot{e}) \equiv \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-4}\, d\varphi \equiv \mathbf{I_{0\text{-}}}(e,\dot{e},n = 3) =$$
$$= \pi[2 + 3(e - \dot{e})^2]\,[1 - (e - \dot{e})^2]^{-7/2}\,.$$

The later equality in the above relation follows from formula (6h) from paper [7], where we have already given the analytical solutions of the integrals (1) – (3) for *integer* values of the power *n* ($n = -1, 0, 1, 2, 3$). The evaluation of the auxiliary integral $\mathbf{A_3}(e,\dot{e})$ requires some additional efforts:

(10) $$\mathbf{A_3}(e,\dot{e}) \equiv \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-3}\, d\varphi = \int_0^{2\pi}\{[1 + (e - \dot{e})cos\varphi] - (e - \dot{e})\cos\varphi\}\times$$
$$\times[1 + (e - \dot{e})cos\varphi]^{-3}\, d\varphi = \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-2}\, d\varphi -$$
$$- (e - \dot{e})\int_0^{2\pi}cos\varphi[1 + (e - \dot{e})cos\varphi]^{-3}\, d\varphi = 2\pi\,[1 - (e - \dot{e})^2]^{-3/2} -$$
$$- (e - \dot{e})\int_0^{2\pi}cos\varphi[1 + (e - \dot{e})cos\varphi]^{-3}\, d\varphi\,,$$

where we have used the mentioned above result (8). To evaluate further the right-hand side of the equality (10), we integrate by parts:

(11) $$\mathbf{A_3}(e,\dot{e}) = 2\pi\,[1 - (e - \dot{e})^2]^{-3/2} - (e - \dot{e})\int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-3}\, d(sin\varphi) =$$

$$= 2\pi\,[1 - (e - \dot{e})^2]^{-3/2} + 3(e - \dot{e})\int_0^{2\pi}(1 - cos^2\varphi)[1 + (e - \dot{e})cos\varphi]^{-4}\,d\varphi =$$

$$= 2\pi\,[1 - (e - \dot{e})^2]^{-3/2} + 3(e - \dot{e})^2\int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-4}\,d\varphi +$$

$$+ 3\int_0^{2\pi}\{[1 - (e - \dot{e})^2cos^2\varphi] - 1\}[1 + (e - \dot{e})cos\varphi]^{-4}\,d\varphi = 2\pi\,[1 - (e - \dot{e})^2]^{-3/2} +$$

$$+ 3(e - \dot{e})^2\mathbf{A_4}(e,\dot{e}) + 3\int_0^{2\pi}[1 - (e - \dot{e})cos\varphi][1 + (e - \dot{e})cos\varphi]^{-3}\,d\varphi -$$

$$- 3\int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-4}\,d\varphi = 2\pi\,[1 - (e - \dot{e})^2]^{-3/2} + 3(e - \dot{e})^2\mathbf{A_4}(e,\dot{e}) + 3\mathbf{A_3}(e,\dot{e}) -$$

$$- 3(e - \dot{e})\int_0^{2\pi}cos\varphi[1 + (e - \dot{e})cos\varphi]^{-3}\,d\varphi - 3\mathbf{A_4}(e,\dot{e})\ .$$

Consequently, we have about the unknown function $\mathbf{A_3}(e,\dot{e})$ that:

$$(12)\qquad - 2\mathbf{A_3}(e,\dot{e}) = 2\pi\,[1 - (e - \dot{e})^2]^{-3/2} + 3[(e - \dot{e})^2 - 1]\mathbf{A_4}(e,\dot{e}) -$$

$$- 3(e - \dot{e})\int_0^{2\pi}cos\varphi[1 + (e - \dot{e})cos\varphi]^{-3}\,d\varphi\ .$$

We can again use the equality (10), but now to write it into a form more appropriate for comparison with (12):

$$(13)\qquad - 2\mathbf{A_3}(e,\dot{e}) = - 4\pi\,[1 - (e - \dot{e})^2]^{-3/2} + 2(e - \dot{e})\int_0^{2\pi}cos\varphi[1 + (e - \dot{e})cos\varphi]^{-3}\,d\varphi\ .$$

Equating of the right-hand-sides of (12) and (13) enables us to compute the unknown integral. Strictly speaking, this is the integral $\mathbf{I_1}(e,\dot{e},n = 2)$ (see the definition (3) for $\boldsymbol{j} = 1$ and $n = 2$):

$$(14)\qquad 5(e - \dot{e})\int_0^{2\pi}cos\varphi[1 + (e - \dot{e})cos\varphi]^{-3}\,d\varphi \equiv 5(e - \dot{e})\mathbf{I_1}(e,\dot{e},n = 2) =$$

$$= 6\pi\,[1 - (e - \dot{e})^2]^{-3/2} - 3[1 - (e - \dot{e})^2]\mathbf{A_4}(e,\dot{e})\ .$$

Dividing this result by ( – 5) and replacing it into the right side of the relation (10), we obtain the expression for the unknown function $\mathbf{A_3}(e,\dot{e})$:

$$(15)\qquad \mathbf{A_3}(e,\dot{e}) = 2\pi\,[1 - (e - \dot{e})^2]^{-3/2} - (6\pi/5)[1 - (e - \dot{e})^2]^{-3/2} +$$
$$+ (3\pi/5)[1 - (e - \dot{e})^2][2 + 3(e - \dot{e})^2][1 - (e - \dot{e})^2]^{-7/2}\ ,$$

where we have applied the already computed expression (9) for $\mathbf{A_4}(e,\dot{e})$. Finally:

$$(16)\qquad \mathbf{A_3}(e,\dot{e}) \equiv \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-3}\, d\varphi = (4\pi/5)[1 - (e - \dot{e})^2]^{-3/2} +$$

$$+ (3\pi/5)[2 + 3(e - \dot{e})^2][1 - (e - \dot{e})^2]^{-5/2} \equiv (\pi/5)[10 + 5(e - \dot{e})^2][1 - (e - \dot{e})^2]^{-5/2} \equiv$$
$$\equiv \pi[2 + (e - \dot{e})^2][1 - (e - \dot{e})^2]^{-5/2} .$$

Of course, we are able to use the equation (14) for obtaining the analytical solution of the integral $\mathbf{I_1}(e,\dot{e},n = 2)$:

$$(17)\qquad \mathbf{I_1}(e,\dot{e},n = 2) \equiv \int_0^{2\pi} cos\varphi[1 + (e - \dot{e})cos\varphi]^{-3}\, d\varphi = [5(e - \dot{e})]^{-1}\{6\pi[1 - (e - \dot{e})^2]^{-3/2} -$$

$$- 3\pi[1 - (e - \dot{e})^2][\, 2 + 3(e - \dot{e})^2][1 - (e - \dot{e})^2]^{-7/2} = - 3\pi(e - \dot{e})[\, 1 - (e - \dot{e})^2]^{-5/2}.$$

This is already derived expression (paper [7], formula (5b)). We rewrite it only to underline the consistency of our computation with the earlier evaluations. Let us now set about the analytical computation of the integral $\mathbf{A_5}(e,\dot{e})$. We shall proceed in the following manner: we begin with a transformation of the already evaluated integral $\mathbf{A_4}(e,\dot{e})$, which leads to an integration by parts. The later operation will require differentiation with respect to $\varphi$ ($\varphi$ is the azimuthal angle, which in our case is the integration variable) of the quantity $[1 + (e - \dot{e})cos\varphi]^{-4}$. As a consequence, in the integrand of the one of the terms appears the factor $[1 + (e - \dot{e})cos\varphi]^{-5}$, generating, as the final result, the integral $\mathbf{A_5}(e,\dot{e})$, which we are seeking for.

$$(18)\qquad \mathbf{A_4}(e,\dot{e}) \equiv \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-4}\, d\varphi = \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi][1 + (e - \dot{e})cos\varphi]^{-4} d\varphi -$$

$$- (e - \dot{e})\int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-4}\, d(sin\varphi) = \int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-3}\, d\varphi -$$

$$- (e - \dot{e})\{sin\varphi[1 + (e - \dot{e})cos\varphi]^{-4}\Big|_0^{2\pi} - 4(e - \dot{e})\int_0^{2\pi} sin^2\varphi[1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi\} =$$

$$= \mathbf{A_3}(e,\dot{e}) + 4(e - \dot{e})^2\int_0^{2\pi}(1 - cos^2\varphi)[1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi = \mathbf{A_3}(e,\dot{e}) +$$

$$+ 4(e - \dot{e})^2\int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi + 4\int_0^{2\pi}\{[1 - (e - \dot{e})^2cos^2\varphi] - 1\}[1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi =$$

$$= \mathbf{A_3}(e,\dot{e}) + 4(e - \dot{e})^2\mathbf{A_5}(e,\dot{e}) + 4\int_0^{2\pi}[1 - (e - \dot{e})cos\varphi][1 + (e - \dot{e})cos\varphi][1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi -$$

$$-4\int_0^{2\pi}[1+(e-\dot{e})cos\varphi]^{-5}\,d\varphi = \mathbf{A_3}(e,\dot{e}) - 4[1-(e-\dot{e})^2]\mathbf{A_5}(e,\dot{e}) + 4\mathbf{A_4}(e,\dot{e}) -$$

$$-4\int_0^{2\pi}\{[1+(e-\dot{e})cos\varphi]-1\}[1+(e-\dot{e})cos\varphi]^{-4}\,d\varphi = \mathbf{A_3}(e,\dot{e}) + 4\mathbf{A_4}(e,\dot{e}) -$$

$$-4[1-(e-\dot{e})^2]\mathbf{A_5}(e,\dot{e}) - 4\mathbf{A_3}(e,\dot{e}) + 4\mathbf{A_4}(e,\dot{e}) =$$
$$= -3\mathbf{A_3}(e,\dot{e}) + 8\mathbf{A_4}(e,\dot{e}) - 4[1-(e-\dot{e})^2]\mathbf{A_5}(e,\dot{e}) .$$

The above result enables us to express $\mathbf{A_5}(e,\dot{e})$ by means of $\mathbf{A_3}(e,\dot{e})$ and $\mathbf{A_4}(e,\dot{e})$, and applying the relations (15) and (9), respectively, to write the final form for this integral:

$$(19)\qquad \mathbf{A_5}(e,\dot{e}) \equiv \int_0^{2\pi}[1+(e-\dot{e})cos\varphi]^{-5}\,d\varphi = \{4[1-(e-\dot{e})^2]\}^{-1}[-3\mathbf{A_3}(e,\dot{e}) + 7\mathbf{A_4}(e,\dot{e})] =$$

$$= \{4[1-(e-\dot{e})^2]\}^{-1}\{-3\pi[2+(e-\dot{e})^2][1-(e-\dot{e})^2]^{-5/2} +$$
$$+ 7\pi[2+3(e-\dot{e})^2][1-(e-\dot{e})^2]^{-7/2}\} = (\pi/4)[1-(e-\dot{e})^2]^{-9/2}[8+24(e-\dot{e})^2 +$$
$$+ 3(e-\dot{e})^4] \equiv$$
$$\equiv (\pi/4)(8 + 24e^2 + 3e^4 - 48e\dot{e} - 12e^3\dot{e} + 24\dot{e}^2 + 18e^2\dot{e}^2 - 12e\dot{e}^3 + 3\dot{e}^4)[1-(e-\dot{e})^2]^{-9/2}.$$

With a view to a further use, we also write the expressions for the integrals $\mathbf{A_i}(e,\dot{e})$, ($\mathbf{i}$ = 1,…, 5), when $\boldsymbol{\dot{e}(u) = 0}$. Geometrically, this situation corresponds to the case, when all particles have orbits with some (constant) eccentricity throughout the considered accretion disc. According to the formulas (7), (8), (16), (9) and (19), we have, respectively:

$$(20)\qquad \mathbf{A_1}(e,0) \equiv \int_0^{2\pi}(1+e\,cos\varphi)^{-1}\,d\varphi = 2\pi\,(1-e^2)^{-1/2} ,$$

$$(21)\qquad \mathbf{A_2}(e,0) \equiv \int_0^{2\pi}(1+e\,cos\varphi)^{-2}\,d\varphi = 2\pi\,(1-e^2)^{-3/2} ,$$

$$(22)\qquad \mathbf{A_3}(e,0) \equiv \int_0^{2\pi}(1+e\,cos\varphi)^{-3}\,d\varphi = \pi(2+e^2)(1-e^2)^{-5/2} ,$$

$$(23)\qquad \mathbf{A_4}(e,0) \equiv \int_0^{2\pi}(1+e\,cos\varphi)^{-4}\,d\varphi = \pi(2+3e^2)(1-e^2)^{-7/2} ,$$

$$(24)\qquad \mathbf{A_5}(e,0) \equiv \int_0^{2\pi}(1+e\,cos\varphi)^{-5}\,d\varphi = (\pi/4)(8+24e^2+3e^4)(1-e^2)^{-9/2} .$$

### 2.2. Evaluation of integrals of the type

$$\mathbf{J_i}(\boldsymbol{e},\boldsymbol{\dot{e}}) \equiv \int_0^{2\pi} (\mathbf{1} + \boldsymbol{e\cos\varphi})^{-1}[\mathbf{1} + (\boldsymbol{e} - \boldsymbol{\dot{e}})\boldsymbol{\cos\varphi}]^{-\mathbf{i}}\, \boldsymbol{d\varphi}$$

The noteworthy for these integrals, which are also functions only of the variables $e(u)$ and $\dot{e}(u)$, is that their denominators are products of the multipliers $(1 + e\cos\varphi)$ and $[1 + (e - \dot{e})\cos\varphi]$. The first of them always presents in the denominator in power one, while the later is risen to a power $i$. We shall be interested in values of $i$ equal to 1, 2, 3 and 4.

$$(25) \qquad \mathbf{J_1}(e,\dot{e}) \equiv \int_0^{2\pi} (1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-1}d\varphi = \int_0^{2\pi}[(1 + e\cos\varphi) - e\cos\varphi] \times$$

$$\times\, (1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-1}d\varphi = \int_0^{2\pi}[(1 + (e - \dot{e})\cos\varphi]^{-1}d\varphi -$$

$$- [e/(e - \dot{e})]\int_0^{2\pi}\{[1 + (e - \dot{e})\cos\varphi)] - 1\}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-1}d\varphi =$$

$$= 2\pi[1 - (e - \dot{e})^2]^{-1/2} - [e/(e - \dot{e})]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}d\varphi +$$

$$+ [e/(e - \dot{e})]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-1}d\varphi =$$

$$= 2\pi[1 - (e - \dot{e})^2]^{-1/2} - 2\pi[e/(e - \dot{e})](1 - e^2)^{-1/2} + [e/(e - \dot{e})]\mathbf{J_1}(e,\dot{e}).$$

Here we have used formula **858.525** from Dwight [6], which, in turn, implies the relations (7) and (20), applied above. Consequently, equation (25) gives that:

$$(26) \qquad \{1 - [e/(e - \dot{e})]\}\mathbf{J_1}(e,\dot{e}) \equiv [- \dot{e}/(e - \dot{e})]\mathbf{J_1}(e,\dot{e}) =$$
$$= 2\pi[1 - (e - \dot{e})^2]^{-1/2} - 2\pi[e/(e - \dot{e})](1 - e^2)^{-1/2}.$$

After multiplication by $[-(e - \dot{e})/\dot{e}]$, we obtain the following result for the integral $\mathbf{J_1}(e,\dot{e})$:

$$(27) \qquad \mathbf{J_1}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi =$$

$$= (2\pi/\dot{e})\{e(1 - e^2)^{-1/2} - (e - \dot{e})[1 - (e - \dot{e})^2]^{-1/2}\}.$$

Two circumstances must be pointed out, concerning the validity of this formula:

($i$)We have supposed that $(e - \dot{e}) \neq 0$. If it is not the case, then from the definition (25) it follows that $\mathbf{J_1}(e,\dot{e} = e) = \mathbf{A_1}(e,0) = 2\pi(1 - e^2)^{-1/2}$ (see the solution (20)). But for $e(u) = \dot{e}(u)$, equation (27) gives the *same* result, i.e., it is valid also for the case $e(u) - \dot{e}(u) = 0$, and the limitation $e(u) - \dot{e}(u) \neq 0$ *makes no sense*.

($ii$) In the relation (27) $\dot{e}(u)$ must not be equal to zero. Of course, if we *directly* set into the definition (25) $\dot{e}(u) = 0$, we easily find that:

$$\mathbf{J_1}(e,\dot{e} = 0) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-2}\, d\varphi = \mathbf{A_2}(e,0) = 2\pi\,(1 - e^2)^{-3/2}, \tag{28}$$

From the other hand, we have the availability to apply in (27) the L'Hospital's rule, in order to evaluate the right-hand-side of this expression, when $\dot{e}(u)$ approaches zero. In particular, it is fulfilled the condition:

$$\lim_{\dot{e}(u) \to 0} \{e(1 - e^2)^{-1/2} - (e - \dot{e})[1 - (e - \dot{e})^2]^{-1/2}\} = 0. \tag{29}$$

Further, computation of the derivative with respect to $\dot{e}$ of the difference into the curly brackets gives (after taking the limit $\dot{e} \to 0$):

$$\begin{aligned}&\lim_{\dot{e}(u) \to 0} \partial/\partial\dot{e}\{e(1 - e^2)^{-1/2} - (e - \dot{e})[1 - (e - \dot{e})^2]^{-1/2}\} = \\ &= \lim_{\dot{e}(u) \to 0} \{[1 - (e - \dot{e})^2]^{-1/2} + (e - \dot{e})^2[1 - (e - \dot{e})^2]^{-3/2}\} = (1 - e^2)^{-3/2}.\end{aligned} \tag{30}$$

Consequently, the above result (30) implies that the L'Hospital's rule, when applied to (27), leads to the same result (28), derived by a *direct* substitution $\dot{e}(u) = 0$ into the original formula for the integral $\mathbf{J_1}(e,\dot{e})$. In that sense, we shall use the expression (27) without checking whether $\dot{e}(u) \neq 0$ or not, having in mind that the L'Hospital's rule ensures a continuous transition through the point $\dot{e}(u) = 0$.

$$\begin{aligned}\mathbf{J_2}(e,\dot{e}) &\equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-2}\, d\varphi = \int_0^{2\pi}[(1 + e\cos\varphi) - e\cos\varphi]\times \\ &\times(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-2}\, d\varphi = \int_0^{2\pi}[1 + (e - \dot{e})\cos\varphi]^{-2}\, d\varphi - \\ &- [e/(e - \dot{e})]\int_0^{2\pi}\{[1 + (e - \dot{e})\cos\varphi)] - 1\}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-2}\, d\varphi =\end{aligned} \tag{31}$$

$$= 2\pi[1 - (e - \dot{e})^2]^{-3/2} - [e/(e - \dot{e})]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi +$$

$$+ [e/(e - \dot{e})]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-2}\, d\varphi =$$

$$= 2\pi[1 - (e - \dot{e})^2]^{-3/2} - [e/(e - \dot{e})]\mathbf{J_1}(e,\dot{e}) + [e/(e - \dot{e})]\mathbf{J_2}(e,\dot{e}).$$

Here we again have used formula **858.535** from Dwight [6] (see also the expression (8) for $\mathbf{A_2}(e,\dot{e})$ above) and the definition (25) for the integral (25). The equation (31) enables us to write an explicit form for $\mathbf{J_2}(e,\dot{e})$:

(32) $$- [\dot{e}/(e - \dot{e})]\mathbf{J_2}(e,\dot{e}) = 2\pi[1 - (e - \dot{e})^2]^{-3/2} - [e/(e - \dot{e})]\mathbf{J_1}(e,\dot{e}),$$

or, after dividing by $[- \dot{e}/(e - \dot{e})]$ (under the condition $\dot{e}(u) \neq 0$), we obtain:

(33) $$\mathbf{J_2}(e,\dot{e}) = - 2\pi\, [(e - \dot{e})/\dot{e}]\, [1 - (e - \dot{e})^2]^{-3/2} + (e/\dot{e})\mathbf{J_1}(e,\dot{e}).$$

We are in a position to apply the result (27) for the integral $\mathbf{J_1}(e,\dot{e})$, having in mind the two remarks, which we have already done about the cases $e(u) - \dot{e}(u) = 0$ and $\dot{e}(u) = 0$:

(34) $$\mathbf{J_2}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-2}\, d\varphi = - 2\pi\, [(e - \dot{e})/\dot{e}]\, [1 - (e - \dot{e})^2]^{-3/2} -$$

$$- 2\pi\, [e(e - \dot{e})/\dot{e}^2]\, [1 - (e - \dot{e})^2]^{-1/2} + 2\pi(e^2/\dot{e}^2)(1 - e^2)^{-1/2} \equiv$$
$$\equiv - 2\pi(e - \dot{e})\dot{e}^{-2}[1 - (e - \dot{e})^2]^{-3/2}(e - e^3 + \dot{e} + 2e^2\dot{e} - e\dot{e}^2) + 2\pi(e^2/\dot{e}^2)(1 - e^2)^{-1/2}.$$

Like the case, considering the integral $\mathbf{J_1}(e,\dot{e})$, we again strike with the problem of the applicability of the expression (34) in the general situation. Namely, when $e(u) - \dot{e}(u) = 0$ and /or $\dot{e}(u) = 0$.

(*i*) If the supposition $e(u) - \dot{e}(u) \neq 0$ is not valid, then from the definition (31) for the integral $\mathbf{J_2}(e,\dot{e})$ it *directly* follows that $\mathbf{J_2}(e,\dot{e} = e) = \mathbf{A_1}(e,0) = 2\pi(1 - e^2)^{-1/2}$ (see (20)). It is evident that for $e(u) = \dot{e}(u)$, the expression (34) gives the same result. Although, during the derivation of (34), it was supposed that $e(u) - \dot{e}(u) \neq 0$, in the final result about $\mathbf{J_2}(e,\dot{e})$ this limitation is not leading to any singular effects.

(*ii*) Also, in the relation (34) $\dot{e}(u)$ *must not vanish*. To avoid this constraint, we may directly set $\dot{e}(u) = 0$ in the definition (31). The integral is not singular and is already evaluated. Concretely:

(35) $$\mathbf{J_2}(e,\dot{e} = 0) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-3}\, d\varphi \equiv \mathbf{A_3}(e,0) = \pi(2 + e^2)(1 - e^2)^{-5/2}.$$

We, of course, are able to ask whether the limit transition $\dot{e}(u) \to 0$ in the formula (34) will make sense, giving the above result (35). The supposition can be checked by applying **two** times of the L'Hospital's rule/theorem. To do the proof in a more compact manner, let us rewrite the final result (34) into the following form:

(36) $\mathbf{J_2}(e,\dot{e}) = 2\pi\dot{e}^{-2}(1-e^2)^{-1/2}\mathbf{B}(e,\dot{e}),$

where the function $\mathbf{B}(e,\dot{e})$, according to (34), is defined as:

(37) $\mathbf{B}(e,\dot{e}) \equiv [1-(e-\dot{e})^2]^{-3/2}\{e^2[1-(e-\dot{e})^2]^{3/2} + (-e^2 + e^4 - 3e^3\dot{e} + \dot{e}^2 + 3e^2\dot{e}^2 - e\dot{e}^3)(1-e^2)^{1/2}\}.$

It is easily seen that $\mathbf{B}(e,\dot{e} = 0) = 0$. The other conditions, needed for the applicability of the L'Hospital's rule with respect to the relations (34) and (36), when $\dot{e}(u) \to 0$, are obviously fulfilled. The first partial derivative of $\mathbf{B}(e,\dot{e})$ with respect to the variable $\dot{e}(u)$ is:

(38) $\partial\mathbf{B}(e,\dot{e})/\partial\dot{e} = (1-e^2)^{1/2}[1-(e-\dot{e})^2]^{-5/2}\dot{e}(2 + e^2 - 2e\dot{e} + \dot{e}^2) \equiv \equiv (1-e^2)^{1/2}[1-(e-\dot{e})^2]^{-5/2}\dot{e}[2 + (e-\dot{e})^2].$

Obviously, the partial derivative $\partial\mathbf{B}(e,\dot{e})/\partial\dot{e}$ vanishes for $\dot{e}(u) = 0$. We again see that the partial derivative with respect to $\dot{e}(u)$ of the denominator in (36) is $2\dot{e}(1-e^2)$ and also vanishes for $\dot{e}(u) = 0$. Nevertheless, the last circumstance does not cause troubles, when the transition $\dot{e}(u) \to 0$ is performed. The availability of the factor $\dot{e}(u)$ both in the dominator and the denominator enables us to *cancel out* it, and the expression becomes free from the singularity at $\dot{e}(u) = 0$. In such a way, the needed condition ($v$) in the formulation of the L'Hospital's theorem (given in chapter **2.1.** above) is successfully fulfilled, and the transition $\dot{e}(u) \to 0$ does not generate a singularity. Therefore:

(39)
$$\mathbf{J_2}(e,\dot{e} = 0) = \lim_{\dot{e}(u)\to 0}\{2\pi[\partial\mathbf{B}(e,\dot{e})/\partial\dot{e}]/\partial[\dot{e}^2(1-e^2)^{1/2}]/\partial\dot{e}\} =$$
$$= \lim_{\dot{e}(u)\to 0}[2\pi(1-e^2)^{1/2}\dot{e}(2 + e^2 - 2e\dot{e} + \dot{e}^2)]/\{[2\dot{e}[1-(e-\dot{e})^2]^{5/2}(1-e^2)^{1/2}\} =$$
$$= \pi(2 + e^2)(1-e^2)^{-5/2}.$$

This result is the same as the relation (35), where is used a *direct* method for computation of the case $\dot{e}(u) = 0$, without the application of any continuum transitions to this special point $\dot{e}(u) = 0$. Consequently, we are able to use the formula (34) also for values $\dot{e}(u) = 0$, keeping in mind that the singularity may be overcome by means of the L'Hospital's rule.

$$(40)\qquad \mathbf{J_3}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e-\dot{e})\cos\varphi]^{-3}\,d\varphi = \int_0^{2\pi}[(1 + e\cos\varphi) - e\cos\varphi]\times$$

$$\times(1 + e\cos\varphi)^{-1}[1 + (e-\dot{e})\cos\varphi]^{-3}\,d\varphi = \int_0^{2\pi}[1 + (e-\dot{e})\cos\varphi]^{-3}\,d\varphi -$$

$$-\,[e/(e-\dot{e})]\int_0^{2\pi}\{[1 + (e-\dot{e})\cos\varphi] - 1\}(1 + e\cos\varphi)^{-1}[1 + (e-\dot{e})\cos\varphi]^{-3}\,d\varphi =$$

$$= \mathbf{A_3}(e,\dot{e}) - [e/(e-\dot{e})]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e-\dot{e})\cos\varphi]^{-2}\,d\varphi +$$

$$+\,[e/(e-\dot{e})]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e-\dot{e})\cos\varphi]^{-3}\,d\varphi =$$

$$= \mathbf{A_3}(e,\dot{e}) - [e/(e-\dot{e})]\mathbf{J_2}(e,\dot{e}) + [e/(e-\dot{e})]\mathbf{J_3}(e,\dot{e}).$$

Transferring the unknown function $\mathbf{J_3}(e,\dot{e})$ into the left-hand-side, we obtain:

$$(41)\qquad [1 - e/(e-\dot{e})]\mathbf{J_3}(e,\dot{e}) \equiv -\,[\dot{e}/(e-\dot{e})]\mathbf{J_3}(e,\dot{e}) = \mathbf{A_3}(e,\dot{e}) - [e/(e-\dot{e})]\mathbf{J_2}(e,\dot{e}).$$

During the present derivation, we are supposing that both $\dot{e}(u) \neq 0$ and $e(u) - \dot{e}(u) \neq 0$. In agreement with this suggestion, we rewrite the above relation as:

$$(42)\qquad \mathbf{J_3}(e,\dot{e}) = -\,[(e-\dot{e})/\dot{e}]\mathbf{A_3}(e,\dot{e}) + (e/\dot{e})\mathbf{J_2}(e,\dot{e}) = -\,\pi(e-\dot{e})[2 + (e-\dot{e})^2]\dot{e}^{-1}\times$$
$$\times[1 - (e-\dot{e})^2]^{-5/2} - 2\pi e(e-\dot{e})(e - e^3 + 2e^2\dot{e} - e\dot{e}^2)\dot{e}^{-3}[1 - (e-\dot{e})^2]^{-3/2} +$$
$$+\,2\pi e^3\dot{e}^{-3}(1-e^2)^{-1/2} =$$
$$= -\,\pi(e-\dot{e})\dot{e}^{-3}[1 - (e-\dot{e})^2]^{-5/2}(2e^2 - 4e^4 + 2e^6 + 2e\dot{e} + 6e^3\dot{e} - 8e^5\dot{e} + 2\dot{e}^2 + e^2\dot{e}^2 +$$
$$+12e^4\dot{e}^2 - 4e\dot{e}^3 - 8e^3\dot{e}^3 + \dot{e}^4 + 2e^2\dot{e}^4) + 2\pi e^3\dot{e}^{-3}(1-e^2)^{-1/2}.$$

We again strike with the problem concerning the validity of the expression (42). If we set into the definition (40) $e(u) = \dot{e}(u)$ (i.e., $e(u) - \dot{e}(u) = 0$), the *direct* computation of the integral $\mathbf{J_3}(e,\dot{e} = e)$ leads to the following simple result (taking into account the equality (20)):

$$(43)\qquad \mathbf{J_3}(e,\dot{e} = e) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-1}\,d\varphi = \mathbf{A_1}(e,0) = 2\pi\,(1-e^2)^{-1/2}\,.$$

The same answer gives equation (42), if the equality $e(u) - \dot{e}(u) = 0$ is set into it. Therefore, the established analytical evaluation (42) for the integral $\mathbf{J_3}(e,\dot{e})$ may be used also in the case when $e(u) - \dot{e}(u) = 0$. This reasoning remains valid even if $e(u) = \dot{e}(u) = 0$, under the condition that $e^3(u)$ and $\dot{e}^3(u)$ are preliminary cancelled out in the last term $2\pi e^3\dot{e}^{-3}(1 - e^2)^{-1/2}$ of the expression (42). Then the result is trivial: $\mathbf{J_3}(0,0) = 2\pi$. Like the previous two cases about the integrals $\mathbf{J_1}(e,\dot{e})$ and $\mathbf{J_2}(e,\dot{e})$, the special case $\dot{e}(u) = 0$ can be resolved by means of a *direct* substitution of this constraint into the definition of the integral $\mathbf{J_3}(e,\dot{e} = 0)$:

$$(44)\qquad \mathbf{J_3}(e,\dot{e} = 0) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-4}\, d\varphi = \mathbf{A_4}(e,0) = \pi\,(2 + 3e^2)(1 - e^2)^{-7/2}\,.$$

Here we have applied the evaluation (23) of the function $\mathbf{A_4}(e,0)$. It may be checked, that in the limit $\dot{e}(u) \to 0$, the expression (42) gives a result that coincides with the last term of the equality (43). For this purpose, the L'Hospital's rule must be come into use again. We shall not perform these tedious calculations in the present paper, which are essentially of the same character, as in the cases of the integrals $\mathbf{J_1}(e,\dot{e})$ and $\mathbf{J_2}(e,\dot{e})$, considered under the transition $\dot{e}(u) \to 0$. We shall only mention, that such a transition gives a continuous result, when is applied to the relation (42). The same approach will be put into use under the computation of the integral $\mathbf{J_4}(e,\dot{e})$, to which we are now going on.

$$(45)\qquad \mathbf{J_4}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-4}\, d\varphi = \int_0^{2\pi}[(1 + e\cos\varphi) - e\cos\varphi]\times$$

$$\times(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-4}\, d\varphi = \int_0^{2\pi}[1 + (e - \dot{e})\cos\varphi]^{-4}\, d\varphi -$$

$$- [e/(e - \dot{e})]\int_0^{2\pi}\{[1 + (e - \dot{e})\cos\varphi] - 1\}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-4}\, d\varphi =$$

$$= \mathbf{A_4}(e,\dot{e}) - [e/(e - \dot{e})]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-3}\, d\varphi +$$

$$+ [e/(e - \dot{e})]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-4}\, d\varphi =$$

$$= \mathbf{A_4}(e,\dot{e}) - [e/(e - \dot{e})]\mathbf{J_3}(e,\dot{e}) + [e/(e - \dot{e})]\mathbf{J_4}(e,\dot{e}).$$

We have applied above the definitions (9), (40) and the first equality/identity in (45). Supposing that both $\dot{e}(u) \neq 0$ and $e(u) - \dot{e}(u) \neq 0$, we proceed further to evaluate the integral $\mathbf{J_4}(e,\dot{e})$ through the already computed functions of $e(u)$ and $\dot{e}(u)$ $\mathbf{A_4}(e,\dot{e})$ and $\mathbf{J_3}(e,\dot{e})$:

$$(46)\qquad [1 - e/(e - \dot{e})]\mathbf{J_4}(e,\dot{e}) \equiv - [\dot{e}/(e - \dot{e})]\mathbf{J_4}(e,\dot{e}) = \mathbf{A_4}(e,\dot{e}) - [e/(e - \dot{e})]\mathbf{J_3}(e,\dot{e}).$$

Consequently, on the basis of the relations (9) for $\mathbf{A_4}(e,\dot{e})$ and (42) for $\mathbf{J_3}(e,\dot{e})$, we obtain the final expression for $\mathbf{J_4}(e,\dot{e})$ in an explicit form in terms of $e(u)$ and $\dot{e}(u)$:

$$(47)\quad \mathbf{J_4}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-4}\, d\varphi = - [(e - \dot{e})/\dot{e}]\mathbf{A_4}(e,\dot{e}) + (e/\dot{e})\mathbf{J_3}(e,\dot{e}) =$$

$$\begin{aligned}
&= - (e - \dot{e})[2 + 3(e - \dot{e})^2]\dot{e}^{-1}[1 - (e - \dot{e})^2]^{-7/2} - \pi e(e - \dot{e})( 2e^2 - 4e^4 + 2e^6 + 2e\dot{e} + \\
&+ 6e^3\dot{e} - 8e^5\dot{e} + 2\dot{e}^2 + e^2\dot{e}^2 + 12e^4\dot{e}^2 - 4e\dot{e}^3 - 8e^3\dot{e}^3 + \dot{e}^4 + 2e^2\dot{e}^4)\dot{e}^{-4}[1 - (e - \dot{e})^2]^{-5/2} + \\
&+ 2\pi e^4\dot{e}^{-4}(1 - e^2)^{-1/2} \equiv \\
&\equiv \pi(- 2e^4 + 6e^6 - 6e^8 + 2e^{10} - 14e^5\dot{e} + 28e^7\dot{e} - 14e^9\dot{e} + 7e^4\dot{e}^2 - 49e^6\dot{e}^2 + 42e^8\dot{e}^2 + \\
&+ 35e^5\dot{e}^3 - 70e^7\dot{e}^3 + 2\dot{e}^4 + 8e^2\dot{e}^4 + 70e^6\dot{e}^4 - 10e\dot{e}^5 - 14e^3\dot{e}^5 - 42e^5\dot{e}^5 + 3\dot{e}^6 + 7e^2\dot{e}^6 + \\
&+ 14e^4\dot{e}^6 - e\dot{e}^7 - 2e^3\dot{e}^7) + 2\pi e^4\dot{e}^{-4}(1 - e^2)^{-1/2}.
\end{aligned}$$

Although the above expression is derived under the restriction $e(u) - \dot{e}(u) \neq 0$, it makes sense even if $e(u) - \dot{e}(u) = 0$. In the last case, the relation (47) shows that:

$$(48)\quad \mathbf{J_4}(e,\dot{e} = e) = 2\pi e^4\dot{e}^{-4}(1 - e^2)^{-1/2} = 2\pi(1 - e^2)^{-1/2},$$

because, obviously, the first two terms in the right-hand-side are equal to zero for $e(u) - \dot{e}(u) = 0$, and remains only the last term, where $e(u)/\dot{e}(u) = 1$. Of course, the integral $\mathbf{J_4}(e,\dot{e} = e)$ may be *directly* computed by setting $e(u) = \dot{e}(u)$ into its definition (45):

$$(49)\qquad \mathbf{J_4}(e,\dot{e} = e) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-1}\, d\varphi = \mathbf{A_1}(e,0) = 2\pi(1 - e^2)^{-1/2},$$

where we have again used the relation (20). The two expressions (48) and (49) coincide, and, therefore, the restriction $e(u) - \dot{e}(u) \neq 0$ for the solution (47) can be removed. This conclusion continues to be valid even if $\dot{e}(u) = 0$. Concerning the general case $\dot{e}(u) = 0$, the definition (45) also *directly* enables us to evaluate the wanted function $\mathbf{J_4}(e,0)$, namely:

$$(50)\qquad \mathbf{J_4}(e,0) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-5}\, d\varphi \equiv \mathbf{A_5}(e,0) = (\pi/4)(8 + 24e^2 + 3e^4)(1 - e^2)^{-9/2},$$

where we have applied the relation (24). It is possible to check, by means of the L'Hospital's rule for revealing of indeterminacies of the type 0/0, that the passage to the limit $\dot{e}(u) = 0$ in the relation (47) leads to the same result (50). Consequently, such a transition through the point $\dot{e}(u) = 0$ is continuous. Further we shall apply formula (47) also when $\dot{e}(u) = 0$, having in mind that the indeterminacy is overcome preliminary through the

L'Hospital's rule. Like in the previous case for the integral $\mathbf{J_4}(e,\dot{e})$, we shall skip, for reasons of brevity, the proof of the statement that the transition $\dot{e}(u) \to 0$ in (47) gives the same result as the relation (50).

In conclusion, we note that the considered number of integrals of the type $\mathbf{J_i}(e,\dot{e})$, ($\boldsymbol{i}$ = 1, 2, 3, 4), is enough for our consequent applications. They will be made in the forthcoming papers (in particular, paper [8]), devoted to the simplification of the dynamical equation of the accretion discs with elliptical shapes. Summarizing some of the results in this chapter, we mention that for all $\boldsymbol{i}$ = 1, 2, 3 and 4 we have $\mathbf{J_i}(e,\dot{e} = 0) = \mathbf{A_{i+1}}(e,\dot{e} = 0)$, and $\mathbf{J_i}(e,\dot{e} = e) = \mathbf{A_1}(e,\dot{e} = 0) = 2\pi(1 - e^2)^{-1/2}$.

### 2.3. Evaluation of integrals of the type

$$\mathbf{H_i}(\boldsymbol{e},\dot{\boldsymbol{e}}) \equiv \int_0^{2\pi}(\mathbf{1} + \boldsymbol{e}\,\boldsymbol{cos\varphi})^{-\mathbf{i}}[\mathbf{1} + (\boldsymbol{e} - \dot{\boldsymbol{e}})\boldsymbol{cos\varphi}]^{-\mathbf{1}}\, \boldsymbol{d\varphi}$$

These auxiliary integrals will be evaluated analytically for values of the power $\boldsymbol{i}$ = 1, 2, 3 and 4. Their estimates will be applied, in own turn, for computation of *other* auxiliary integrals, which will be made in subsequent papers. We begin with the most simple of them, namely, the integral $\mathbf{H_1}(e,\dot{e})$:

$$(51)\qquad \mathbf{H_1}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi \equiv \mathbf{J_1}(e,\dot{e}) =$$

$$= (2\pi/\dot{e})\{e(1 - e^2)^{-1/2} - (e - \dot{e})[1 - (e - \dot{e})^2]^{-1/2}\},$$

where we rewrite formula (27) above. All remarks, whish were made about the validity of the estimation (27) for the integral $\mathbf{J_1}(e,\dot{e})$, automatically remain in power also for $\mathbf{H_1}(e,\dot{e})$. The next step is to find the integral $\mathbf{H_2}(e,\dot{e})$ as a function of its arguments $e(u)$ and $\dot{e}(u)$:

$$(52)\qquad \mathbf{H_2}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-2}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \int_0^{2\pi}\{[1 + (e - \dot{e})\cos\varphi] -$$

$$- (e - \dot{e})\cos\varphi\}(1 + e\cos\varphi)^{-2}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \int_0^{2\pi}(1 + e\cos\varphi)^{-2}\, d\varphi -$$

$$- [(e - \dot{e})/e]\int_0^{2\pi}[(1 + e\cos\varphi) - 1](1 + e\cos\varphi)^{-2}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \mathbf{A_2}(e,0) -$$

$$- [(e - \dot{e})/e]\int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi + [(e - \dot{e})/e]\int_0^{2\pi}(1 + e\cos\varphi)^{-2}\times$$

$$\times[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \mathbf{A_2}(e,0) - [(e - \dot{e})/e]\mathbf{H_1}(e,\dot{e}) + [(e - \dot{e})/e]\mathbf{H_2}(e,\dot{e}) =$$

$$= 2\pi (1 - e^2)^{-3/2} - 2\pi(e - \dot{e})\dot{e}^{-1}(1 - e^2)^{-1/2} + 2\pi(e - \dot{e})^2 e^{-1}\dot{e}^{-1}[1 - (e - \dot{e})^2]^{-1/2} + \\ + [(e - \dot{e})/e]\mathbf{H_2}(e,\dot{e}),$$

where we have applied the above already established results (21) and (51) $\mathbf{A_2}(e,0)$ and $\mathbf{H_1}(e,\dot{e})$, respectively. Therefore, the equation (52) ensures the following solution for the wanted function $\mathbf{H_2}(e,\dot{e})$:

$$[1 - (e - \dot{e})/e]\mathbf{H_2}(e,\dot{e}) \equiv (\dot{e}/e)\mathbf{H_2}(e,\dot{e}) = 2\pi\{e\dot{e}[1 - (e - \dot{e})^2]^{1/2} - e(e - \dot{e})(1 - e^2)\times \\ \times[1 - (e - \dot{e})^2]^{1/2} + (e - \dot{e})^2(1 - e^2)^{3/2}\}e^{-1}\dot{e}^{-1}(1 - e^2)^{-3/2}[1 - (e - \dot{e})^2]^{-1/2}. \tag{53}$$

After multiplying the both sides of this relation by $e(u)/\dot{e}(u)$, we obtain the final analytical expression for the integral $\mathbf{H_2}(e,\dot{e})$:

$$\mathbf{H_2}(e,\dot{e}) = 2\pi\{(e - \dot{e})^2(1 - e^2)^{3/2} + (- e^2 + e^4 + 2e\dot{e} - e^3\dot{e})[1 - (e - \dot{e})^2]^{1/2}\}\dot{e}^{-2}\times \\ \times(1 - e^2)^{-3/2}[1 - (e - \dot{e})^2]^{-1/2}. \tag{54}$$

Of course, this result is derived under the assumptions that $[e(u) \neq 0] \cap [\dot{e}(u) \neq 0]$. It can be rewritten also into the form:

$$\mathbf{H_2}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-2}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \tag{55}$$

$$= (2\pi/\dot{e}^2)\{(e - \dot{e})^2[1 - (e - \dot{e})^2]^{-1/2} + (- e^2 + e^4 + 2e\dot{e} - e^3\dot{e})(1 - e^2)^{-3/2}\}.$$

Obviously, the formulas (54) and (55) have nonsingular meaning for $e(u) = 0$ (preserving the restriction $\dot{e}(u) \neq 0$), namely:

$$\mathbf{H_2}(0,\dot{e}) = (2\pi/\dot{e}^2)[\dot{e}^2(1 - \dot{e}^2)^{-1/2} + 0] \equiv 2\pi(1 - \dot{e}^2)^{-1/2}. \tag{56}$$

A *direct* computation for the case $e(u) = 0$ (with $\dot{e}(u) \neq 0$) for the integral $\mathbf{H_2}(e = 0,\dot{e})$ shows that:

$$\mathbf{H_2}(0,\dot{e}) \equiv \int_0^{2\pi}(1 - \dot{e}\cos\varphi)^{-1}\, d\varphi = 2\pi(1 - \dot{e}^2)^{-1/2}, \tag{57}$$

which coincides with (56). Here we have again used formula **858.525** from Dwight [6]. Of course, there is not problem to apply the expressions (56) and (57) when $\dot{e}(u) = 0$. They both give the right answer $\mathbf{H_2}(0,0) = 2\pi$.

With respect to the general case $\dot{e}(u) = 0$ (when $e(u)$ does not need to be equal to zero), it may be noted that the expression in the curly brackets in (55) approaches zero, when $\dot{e}(u)$ also approaches zero:

$$\lim_{\dot{e}(u) \to 0}\{(e - \dot{e})^2[1 - (e - \dot{e})^2]^{-1/2} + (- e^2 + e^4 + 2e\dot{e} - e^3\dot{e})(1 - e^2)^{-3/2}\} = \\ = e^2(1 - e^2)^{-1/2} - e^2(1 - e^2)(1 - e^2)^{-3/2} = 0. \tag{58}$$

It is easily verified, that the other conditions for the applicability of the L'Hospital's theorem are also fulfilled. Consequently, we are in a position to apply the L'Hospital's rule with regard to the right-hand-side of the relation (55), in order to overcome the indeterminacy of the type 0/0, when $\dot{e}(u) = 0$. In that approach, we have to evaluate the limit transition:

$$(59)\qquad \lim_{\dot{e}(u)\to 0} \partial/\partial\dot{e}\{(e-\dot{e})^2[1-(e-\dot{e})^2]^{-1/2} + (-e^2+e^4+2e\dot{e}-e^3\dot{e})(1-e^2)^{-3/2}\} =$$
$$= \lim_{\dot{e}(u)\to 0}\{-2(e-\dot{e})[1-(e-\dot{e})^2]^{-1/2} - (e-\dot{e})^3[1-(e-\dot{e})^2]^{-3/2} +$$
$$+ (2e-e^3)(1-e^2)^{-3/2}\} = -2e(1-e^2)^{-1/2} - e^3(1-e^2)^{-3/2} +$$
$$+ (2e-e^3)(1-e^2)^{-3/2} = 0.$$

The computation of the expression (55) in the limit $\dot{e}(u) \to 0$ again strikes with the problem of evaluating of an indeterminacy of the type 0/0. To solve the task, we shall use for a second time the L'Hospital's rule. The premises to do this are available. In particular, we see that:

$$(60)\qquad \lim_{\dot{e}(u)\to 0} \partial/\partial\dot{e}\{-2(e-\dot{e})[1-(e-\dot{e})^2]^{-1/2} - (e-\dot{e})^3[1-(e-\dot{e})^2]^{-3/2} +$$
$$+ (2e-e^3)(1-e^2)^{-3/2}\} = \lim_{\dot{e}(u)\to 0}\{2[1-(e-\dot{e})^2]^{-1/2} + 2(e-\dot{e})^2[1-(e-\dot{e})^2]^{-3/2} +$$
$$+ 3(e-\dot{e})^2[1-(e-\dot{e})^2]^{-3/2} + 3(e-\dot{e})^4[1-(e-\dot{e})^2]^{-5/2}\} =$$
$$= 2(1-e^2)^{-1/2} + 5e^2(1-e^2)^{-3/2} + 3e^4(1-e^2)^{-5/2} = (2+e^2)(1-e^2)^{-5/2}.$$

Consequently, the twice recurrent application of the L'Hospital's rule with respect to the right-hand-side of the equation (55), leads to the following result, when $\dot{e}(u)$ approaches zero value:

$$(61)\qquad \mathbf{H_2}(e,0) = \lim_{\dot{e}(u)\to 0}\mathbf{\{}(2\pi/\dot{e}^2)\{(e-\dot{e})^2[1-(e-\dot{e})^2]^{-1/2} +$$
$$+ (-e^2+e^4+2e\dot{e}-e^3\dot{e})(1-e^2)^{-3/2}\}\mathbf{\}} = \pi(2+e^2)(1-e^2)^{-5/2}.$$

From the other hand, the *direct* calculation for $\dot{e}(u) = 0$ leads to (according to the relation (22)):

$$(62)\qquad \mathbf{H_2}(e,0) \equiv \int_0^{2\pi}(1+e\cos\varphi)^{-3}\, d\varphi \equiv \mathbf{A_3}(e,0) = \pi(2+e^2)(1-e^2)^{-5/2},$$

which coincides with the previous equality (61). In this connection, we note that the transition $\dot{e}(u) \to 0$ in (54) and (55) is continuous. That is to say, when we use the later two formulas for $\dot{e}(u) = 0$, we shall subtend the meaning $\pi(2 + e^2)(1 - e^2)^{-5/2}$. Now it is trivial to evaluate $\mathbf{H_2}(e,\dot{e})$ when both $e(u) = 0$ and $\dot{e}(u) = 0$:

(63) $$\mathbf{H_2}(0,0) \equiv \int_0^{2\pi} d\varphi = 2\pi .$$

The same result follows if we set in (56) $\dot{e}(u) = 0$, or if we set in (61) $e(u) = 0$. This implies that there is not matter the order of the performing of the transitions $e(u) \to 0$ or $\dot{e}(u) \to 0$. The next integral, of the considered in the present paragraph type, is $\mathbf{H_3}(e,\dot{e})$ :

(64) $$\mathbf{H_3}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-3}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \int_0^{2\pi}\{[1 + (e - \dot{e})\cos\varphi] -$$

$$- (e - \dot{e})\cos\varphi\}(1 + e\cos\varphi)^{-3}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \int_0^{2\pi}(1 + e\cos\varphi)^{-3}\, d\varphi -$$

$$- [(e - \dot{e})/e]\int_0^{2\pi}[(1 + e\cos\varphi) - 1](1 + e\cos\varphi)^{-3}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \mathbf{A_3}(e,0) -$$

$$- [(e - \dot{e})/e]\int_0^{2\pi}(1 + e\cos\varphi)^{-2}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi + [(e - \dot{e})/e]\int_0^{2\pi}(1 + e\cos\varphi)^{-3}\times$$

$$\times[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \mathbf{A_3}(e,0) - [(e - \dot{e})/e]\mathbf{H_2}(e,\dot{e}) + [(e - \dot{e})/e]\mathbf{H_3}(e,\dot{e}).$$

After taking into account the expressions (22) and (55) for $\mathbf{A_3}(e,0)$ and $\mathbf{H_2}(e,\dot{e})$, respectively, the unknown function $\mathbf{H_3}(e,\dot{e})$ may be find in an explicit form:

(65) $$\begin{aligned}&[1 - (e - \dot{e})/e]\mathbf{H_3}(e,\dot{e}) \equiv (\dot{e}/e)\mathbf{H_3}(e,\dot{e}) = \pi(2 + e^2)(1 - e^2)^{-5/2} -\\ &- 2\pi(e - \dot{e})^3 e^{-1}\dot{e}^{-2}[1 - (e - \dot{e})^2]^{-1/2} -\\ &- 2\pi(e - \dot{e})(- e^2 + e^4 + 2e\dot{e} - e^3\dot{e})e^{-1}\dot{e}^{-2}(1 - e^2)^{-3/2}.\end{aligned}$$

After multiplying this equation by $e(u)/\dot{e}(u)$ and some other simplifications, we obtain:

(66) $$\begin{aligned}&\mathbf{H_3}(e,\dot{e}) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-3}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \pi e(2e^2 - 4e^4 + 2e^6 - 6e\dot{e} +\\ &+ 10e^3\dot{e} - 4e^5\dot{e} + 6\dot{e}^2 - 5e^2\dot{e}^2 + 2e^4\dot{e}^2)\dot{e}^{-3}(1 - e^2)^{-5/2} -\\ &- 2\pi(e - \dot{e})^2\, \dot{e}^{-3}[1 - (e - \dot{e})^2]^{-1/2} \equiv\\ &\equiv \pi\{(2e^3 - 4e^5 + 2e^7 - 6e^2\dot{e} + 10e^4\dot{e} - 4e^6\dot{e} + 6e\dot{e}^2 - 5e^3\dot{e}^2 + 2e^5\dot{e}^2)[1 - (e - \dot{e})^2]^{1/2} -\\ &- \quad 2(e - \dot{e})^3(1 - e^2)^{5/2}\}\dot{e}^{-3}(1 - e^2)^{-5/2}[1 - (e - \dot{e})^2]^{-1/2}.\end{aligned}$$

It is evident from the above derivation, that the solution (66) is determined under the suggestion that both $e(u) \neq 0$ and $\dot{e}(u) \neq 0$. The first restriction $e(u) \neq 0$ may be eliminated, if we note that the right-hand-side of (66) makes sense even if we set into it $e(u) = 0$, preserving the other condition $\dot{e}(u) \neq 0$:

$$(67)\qquad \mathbf{H_3}(0,\dot e) \equiv \int_0^{2\pi}(1 - \dot e cos\varphi)^{-1}\, d\varphi = 2\pi \dot e^3 \dot e^{-3}(1 - \dot e^2)^{-1/2} \equiv 2\pi(1 - \dot e^2)^{-1/2}.$$

Of course, the non-vanishing of $\dot e(u)$ ensures the possibility to cancel out the factor $\dot e^3(u)$, which presents into the nominator and the denominator of the above quotient. The same result may be established, if we set *directly* $e(u) = 0$ into the definition (64)of the integral $\mathbf{H_3}(e,\dot e)$, and apply the already known relation (20) for the integral $\mathbf{A_1}(\dot e,0)$:

$$(68)\qquad \mathbf{H_3}(0,\dot e) \equiv \int_0^{2\pi}(1 - \dot e cos\varphi)^{-1}\, d\varphi \equiv \mathbf{A_1}(\dot e,0) = 2\pi(1 - \dot e^2)^{-1/2}.$$

In this manner, we conclude that the introduced during the derivation of equation (66), restriction $e(u) \neq 0$ is not burdensome. The final result (66) nevertheless gives the right answer, if we formally set into it the “peculiar” value $e(u) = 0$. A little more difficult is the problem concerning the other restriction $\dot e(u) \neq 0$. To consider this case in a compact form, let us introduce the notation $\mathbf{C}(e,\dot e)$ about the term into the curly brackets in the relation (66):

$$(69)\qquad \mathbf{C}(e,\dot e) \equiv (2e^3 - 4e^5 + 2e^7 - 6e^2\dot e + 10e^4\dot e - 4e^6\dot e + 6e\dot e^2 - 5e^3\dot e^2 + 2e^5\dot e^2)\times$$
$$\times[1 - (e - \dot e)^2]^{1/2} - 2(e - \dot e)^3(1 - e^2)^{5/2}.$$

Then we rewrite (66) into the following way:

$$(70)\qquad \mathbf{H_3}(e,\dot e) = \pi\mathbf{C}(e,\dot e)\dot e^{-3}(1 - e^2)^{-5/2}[1 - (e - \dot e)^2]^{-1/2}.$$

Temporally we disregard the factor $\pi(1 - e^2)^{-5/2}[1 - (e - \dot e)^2]^{-1/2}$, which does not cause troubles for $\dot e(u) = 0$, and concentrate on the quotient $\mathbf{C}(e,\dot e)/\dot e^3$. If the later has a reasonable meaning under the limit transition $\dot e(u) \to 0$, then the total product (70) is also defined – it is evaluated simply by multiplication with $\pi(1 - e^2)^{-3}$. Obviously, for $\dot e(u) = 0$, we have $\mathbf{C}(e,0) = 0$, and the other conditions for applying of the L’Hospital’s theorem (when $\dot e(u) \to 0$) are fulfilled too. Computation of the limit $\lim_{\dot e(u) \to 0} \partial\mathbf{C}(e,\dot e)/\partial\dot e$ gives a zero result:

$$(71)\qquad \lim_{\dot e(u) \to 0} \partial\mathbf{C}(e,\dot e)/\partial\dot e = \lim_{\dot e(u) \to 0}\{(- 6e^2 + 10e^4 - 4e^6 + 12e\dot e - 10e^3\dot e + 4e^5\dot e)[1 - (e - \dot e)^2]^{1/2} +$$
$$+ (2e^3 - 4e^5 + 2e^7 - 6e^2\dot e + 10e^4\dot e - 4e^6\dot e + 6e\dot e^2 - 5e^3\dot e^2 + 2e^5\dot e^2)(e - \dot e)\times$$
$$\times[1 - (e - \dot e)^2]^{-1/2} + 6(e - \dot e)^2(1 - e^2)^{5/2}\} = (- 6e^2 + 10e^4 - 4e^6)(1 - e^2)^{1/2} +$$
$$+ e(2e^3 - 4e^5 + 2e^7)(1 - e^2)^{-1/2} + 6e^2(1 - e^2)^{5/2} \equiv 0.$$

The derivative of the denominator with respect to $\dot e(u)$ is $3\dot e^2(u)$, which approaches zero, when $\dot e(u) \to 0$. The conditions for application of the L’Hospital’s rule for computation of $\lim_{\dot e(u) \to 0}[(1/3\dot e^2)\partial\mathbf{C}(e,\dot e)/\partial\dot e]$ are again available, and we have:

$$(72)\quad \begin{aligned}
&\lim_{\dot{e}(u)\to 0} \partial^2\mathbf{C}(e,\dot{e})/\partial\dot{e}^2 = \lim_{\dot{e}(u)\to 0}\{(12e - 10e^3 + 4e^5)[1-(e-\dot{e})^2]^{1/2} + (-6e^2 + 10e^4 - 4e^6 +\\
&+ 12e\dot{e} - 10e^3\dot{e} + 4e^5\dot{e})(e-\dot{e})[1-(e-\dot{e})^2]^{-1/2} + (-8e^3 + 14e^5 - 6e^7 + 24e^2\dot{e} -\\
&- 30e^4\dot{e} + 12e^6\dot{e} - 18e\dot{e}^2 + 15e^3\dot{e}^2 - 6e^5\dot{e}^2)[1-(e-\dot{e})^2]^{-1/2} - (2e^4 - 4e^6 + 2e^8 -\\
&- 8e^3\dot{e} + 14e^5\dot{e} - 6e^7\dot{e} + 12e^2\dot{e}^2 - 15e^4\dot{e}^2 + 6e^6\dot{e}^2 - 6e\dot{e}^3 + 5e^3\dot{e}^3 - 2e^5\dot{e}^3)(e-\dot{e})\times\\
&\times[1-(e-\dot{e})^2]^{-3/2} - 12(e-\dot{e})(1-e^2)^{5/2}\} =\\
&= \lim_{\dot{e}(u)\to 0}\{(12e - 48e^3 + 72e^5 - 48e^7 + 12e^9 + 90e^2\dot{e} - 198e^4\dot{e} + 156e^6\dot{e} - 48e^8\dot{e} - 54e\dot{e}^2 +\\
&+ 225e^3\dot{e}^2 - 198e^5\dot{e}^2 + 72e^7\dot{e}^2 - 132e^2\dot{e}^3 + 120e^4\dot{e}^3 - 48e^6\dot{e}^3 + 36e\dot{e}^4 - 30e^3\dot{e}^4 +\\
&+ 12e^5\dot{e}^4)[1-(e-\dot{e})^2]^{-3/2} - 12(e-\dot{e})(1-e^2)^{5/2}\} = (12e - 48e^3 + 72e^5 - 48e^7 +\\
&+ 12e^9 - 12e + 48e^3 - 72e^5 + 48e^7 - 12e^9)(1-e^2)^{-3/2} \equiv 0.
\end{aligned}$$

Now we are in a position to use the L'Hospital's rule for a third time during the procedure of the evaluation of the solution (66) under the transition $\dot{e}(u) \to 0$. Skipping some of the tedious intermediate algebraic computations, we can write:

$$(73)\quad \begin{aligned}
&\lim_{\dot{e}(u)\to 0} \partial^3\mathbf{C}(e,\dot{e})/\partial\dot{e}^3 = (12e - 48e^3 + 72e^5 - 48e^7 + 12e^9)(1-e^2)^{-5/2} + (90e^2 - 198e^4 +\\
&+ 156e^6 - 48e^8)(1-e^2)^{-3/2} + 12(1-e^2)^{5/2} = 6(1-e^2)^2(2+3e^2).
\end{aligned}$$

This time we obtain a non-zero result, and more importantly, $\partial^3(\dot{e}^3)/\partial\dot{e}^3 = 6 \neq 0$. Having also in mind, that for the first factor in the expression (70) we have:

$$(74)\quad \lim_{\dot{e}(u)\to 0}\{\pi(1-e^2)^{-5/2}[1-(e-\dot{e})^2]^{-1/2}\} = \pi(1-e^2)^{-3},$$

finally, we are able to summarize the following result:

$$(75)\quad \begin{aligned}
&\lim_{\dot{e}(u)\to 0}\mathbf{H_3}(e,\dot{e}) = \lim_{\dot{e}(u)\to 0}\{\pi\mathbf{C}(e,\dot{e})\dot{e}^{-3}(1-e^2)^{-5/2}[1-(e-\dot{e})^2]^{-1/2}\} =\\
&= \pi(2+3e^2)(1-e^2)^{-7/2}.
\end{aligned}$$

There is not a problem to evaluate *directly* the integral $\mathbf{H_3}(e,\dot{e})$ for the special case $\dot{e}(u) = 0$ without making any transition to this value into an expression of the type (66) (or (70), respectively), which is obtained under the preliminary elimination of this case. Therefore, the direct substitution $\dot{e}(u) = 0$ into the definition (66) leads to (taking into account the already known result (23) for $\mathbf{A_4}(e,0)$):

$$(76)\quad \mathbf{H_3}(e,0) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-4}\, d\varphi \equiv \mathbf{A_4}(e,0) = \pi(2+3e^2)(1-e^2)^{-7/2}.$$

The coincidence of the right-hand-side of the formulas (75) and (76) implies that the analytical evaluation (66) for the integral $\mathbf{H_3}(e,\dot{e})$ ensures a *continuous* transition through the “peculiar” value $\dot{e}(u) = 0$. Namely, this property of the analytical derivation (66) will be implicitly understood, when it will be used in the applications. Without specifying whether $\dot{e}(u)$ is equal to zero or not. The same remark concerns the situation $e(u) = 0$ or $e(u) \neq 0$, and also the combination $[e(u) = 0] \cap [e(u) \neq 0]$. In the later, all established above expressions give the right value $\mathbf{H_3}(0,0) = 2\pi$.

The next integral $\mathbf{H_4}(e,\dot{e})$, which we shall try to compute analytically, is the last in the series of integrals of the type, considered in the present subsection. In particular, this is stipulated by the circumstance that, in fact, this is the integral $\mathbf{I_{n\text{-}2\,,\,n+3}}(e,\dot{e},n = -2)$ for the concrete value power $n = -2$. We apply here a notation, which will be put in use in forthcoming papers, where we shall adopt another system of designations for the considered integrals. This integral $\mathbf{H_4}(e,\dot{e})$ participates in an explicit form into the Wronski determinant, establishing the linear dependence/independence between the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$.

(77) $$\mathbf{H_4}(e,\dot{e}) \equiv \mathbf{I_{n\text{-}2\,,\,n+3}}(e,\dot{e},n = -2) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-4}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi =$$

$$= \int_0^{2\pi}\{[1 + (e - \dot{e})\cos\varphi] - (e - \dot{e})\cos\varphi\}(1 + e\cos\varphi)^{-4}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi =$$

$$= \int_0^{2\pi}(1 + e\cos\varphi)^{-4}\, d\varphi - [(e - \dot{e})/e]\int_0^{2\pi}[(1 + e\cos\varphi) - 1](1 + e\cos\varphi)^{-4}\times$$

$$\times[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi = \mathbf{A_4}(e,0) - [(e - \dot{e})/e]\int_0^{2\pi}(1 + e\cos\varphi)^{-3}\times$$

$$\times[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi + [(e - \dot{e})/e]\int_0^{2\pi}(1 + e\cos\varphi)^{-4}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi =$$

$$= \mathbf{A_4}(e,0) - [(e - \dot{e})/e]\mathbf{H_3}(e,\dot{e}) + [(e - \dot{e})/e]\mathbf{H_4}(e,\dot{e}).$$

Taking into account the expressions (23) and (66) for $\mathbf{A_4}(e,0)$ and $\mathbf{H_3}(e,\dot{e})$, respectively, we are able to resolve the above equation (77) with respect to the unknown function $\mathbf{H_4}(e,\dot{e})$ of the variables $e(u)$ and $\dot{e}(u)$. We transfer $\mathbf{H_4}(e,\dot{e})$ into the left-hand side, and taking notice of the of the equality:

(78) $$[1 - (e - \dot{e})/e]\mathbf{H_4}(e,\dot{e}) \equiv (\dot{e}/e)\mathbf{H_4}(e,\dot{e}),$$

we write the following analytical solution:

$$(79)\qquad \mathbf{H_4}(e,\dot{e}) \equiv \mathbf{I_{n-2\,,\,n+3}}(e,\dot{e},n = -2) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-4}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi =$$

$$\begin{aligned}
&= \pi e\dot{e}^2(2 + 3e^2)\dot{e}^{-3}(1 - e^2)^{-7/2} - \pi\dot{e}^{-4}(1 - e^2)^{-5/2}[1 - (e - \dot{e})^2]^{-1/2}\{(e - \dot{e})(2e^3 - \\
&- 4e^5 + 2e^7 - 6e^2\dot{e} + 10e^4\dot{e} - 4e^6\dot{e} + 6e\dot{e}^2 - 5e^3\dot{e}^2 + 2e^5\dot{e}^2)[1 - (e - \dot{e})^2]^{1/2} - \\
&- 2(e - \dot{e})^4(1 - e^2)^{5/2}\} = \pi\dot{e}^{-4}(1 - e^2)^{-7/2}[1 - (e - \dot{e})^2]^{-1/2}\{(2e\dot{e}^3 + 3e^3\dot{e}^3)\times \\
&\times[1 - (e - \dot{e})^2]^{1/2} - (2e^4 - 6e^6 + 6e^8 - 2e^{10} - 8e^3\dot{e} + 22e^5\dot{e} - 20e^7\dot{e} + 6e^9\dot{e} + 12e^2\dot{e}^2 - \\
&- 27e^4\dot{e}^2 + 21e^6\dot{e}^2 - 6e^8\dot{e}^2 - 6e\dot{e}^3 + 11e^3\dot{e}^3 - 7e^5\dot{e}^3 + 2e^7\dot{e}^3)[1 - (e - \dot{e})^2]^{1/2} + \\
&+ 2(e - \dot{e})^4(1 - e^2)^{7/2}\} = \pi\dot{e}^{-4}(1 - e^2)^{-7/2}[1 - (e - \dot{e})^2]^{-1/2}\{(- 2e^4 + 6e^6 - 6e^8 + \\
&+ 2e^{10} + 8e^3\dot{e} - 22e^5\dot{e} + 20e^7\dot{e} - 6e^9\dot{e} - 12e^2\dot{e}^2 + 27e^4\dot{e}^2 - 21e^6\dot{e}^2 + 6e^8\dot{e}^2 + 8e\dot{e}^3 - \\
&- 8e^3\dot{e}^3 + 7e^5\dot{e}^3 - 2e^7\dot{e}^3)[1 - (e - \dot{e})^2]^{1/2} + 2(e - \dot{e})^4(1 - e^2)^{7/2}\} \equiv \\
&\equiv \pi\dot{e}^{-4}(1 - e^2)^{-7/2}(- 2e^4 + 6e^6 - 6e^8 + 2e^{10} + 8e^3\dot{e} - 22e^5\dot{e} + 20e^7\dot{e} - 6e^9\dot{e} - \\
&- 12e^2\dot{e}^2 + 27e^4\dot{e}^2 - 21e^6\dot{e}^2 + 6e^8\dot{e}^2 + 8e\dot{e}^3 - 8e^3\dot{e}^3 + 7e^5\dot{e}^3 - 2e^7\dot{e}^3) + \\
&+ 2\pi(e - \dot{e})^4\dot{e}^{-4}[1 - (e - \dot{e})^2]^{-1/2}.
\end{aligned}$$

Like the previous computations, this solution (79) is derived under the assumptions $[e(u) \neq 0]\cap[\dot{e}(u) \neq 0]$. But nevertheless, it has a definite meaning for $e(u) = 0$. Namely (under a preserving of the restriction $\dot{e}(u) \neq 0$):

$$(80)\qquad \mathbf{H_4}(0,\dot{e}) = 2\pi\dot{e}^4(1 - \dot{e}^2)^{-1/2}\dot{e}^{-4} \equiv 2\pi(1 - \dot{e}^2)^{-1/2}.$$

A *direct* computation (by means of a *direct* substitution $e(u) = 0$ into the definition (77) of the integral $\mathbf{H_4}(e,\dot{e})$) gives the same result as (80):

$$(81)\qquad \mathbf{H_4}(0,\dot{e}) \equiv \int_0^{2\pi}(1 - \dot{e}\cos\varphi)^{-1}\, d\varphi \equiv \mathbf{A_1}(\dot{e},0) = 2\pi(1 - \dot{e}^2)^{-1/2}\ ;$$

(see formula (20) for $\mathbf{A_1}(e,0)$).

Therefore, because the evaluation (81) **does not require** the avoiding of the value $e(u) \neq 0$, we conclude that this restriction is not a factor, which hinders to apply formula (79) in this case. The consideration of the other situation $\dot{e}(u) \neq 0$ requires a more complex treatment, in order to reveal the behaviour of the result (79) under the transition $\dot{e}(u) \to 0$. Having in mind our experience with the previous such problems, we shall try to explore again the L'Hospital's rule. It is appropriate to put to use ***the before*** the **last** expression in the right-hand-side of (79), because the two summands in the **last** expression of (79) **do not separately** satisfy the conditions (*ii*) and (*v*) in the formulation of the L'Hospital's theorem. Concretely: nullification of the limit $\lim_{\dot{e}(u) \to 0} f(e,\dot{e})$ and existing of the limit $\lim_{\dot{e}(u) \to 0} [f\,'(e,\dot{e})/g'(e,\dot{e})]$. In view of that, we define the function $\mathbf{D}(e,\dot{e})$ as follows:

(82) $\mathbf{D}(e,\dot{e}) \equiv (-2e^4 + 6e^6 - 6e^8 + 2e^{10} + 8e^3\dot{e} - 22e^5\dot{e} + 20e^7\dot{e} - 6e^9\dot{e} - 12e^2\dot{e}^2 + 27e^4\dot{e}^2 - 21e^6\dot{e}^2 + 6e^8\dot{e}^2 + 8e\dot{e}^3 - 8e^3\dot{e}^3 + 7e^5\dot{e}^3 - 2e^7\dot{e}^3)[1 - (e - \dot{e})^2]^{1/2} + 2(e - \dot{e})^4(1 - e^2)^{7/2}.$

This definition enables us to rewrite the solution (79) in a more compact form:

(83) $\mathbf{H_4}(e,\dot{e}) \equiv \pi\mathbf{D}(e,\dot{e})\dot{e}^{-4}(1 - e^2)^{-7/2}[1 - (e - \dot{e})^2]^{-1/2}.$

Because

(84) $\lim_{\dot{e}(u) \to 0}\{\pi(1 - e^2)^{-7/2}[1 - (e - \dot{e})^2]^{-1/2}\} = \pi(1 - e^2)^{-4},$

we, as in the previous consideration of $\mathbf{H_3}(e,\dot{e})$, temporally disregard the factor $\pi(1 - e^2)^{-7/2}[1 - (e - \dot{e})^2]^{-1/2}$, and concentrate on the quotient $\mathbf{D}(e,\dot{e})/\dot{e}^4$ under the limit transition $\dot{e}(u) \to 0$. Obviously:

(85) $\mathbf{D}(e,0) = (-2e^4 + 6e^6 - 6e^8 + 2e^{10})(1 - e^2)^{1/2} + 2e^4(1 - e^2)^{7/2} = -2e^4(1 - 3e^2 + 3e^4 - e^6)(1 - e^2)^{1/2} + 2e^4(1 - e^2)^{7/2} = -2e^4(1 - e^2)^3(1 - e^2)^{1/2} + 2e^4(1 - e^2)^{7/2},$

which is a premise to make use of the L'Hospital's rule, in order to investigate if the expression (79) or, equivalently, (83) are well behaved, when $\dot{e}(u)$ approaches zero. We shall not enter in details of the needed (to some extend) tedious algebraic and differential calculations, and only give here some of the final results. For example, it may be shown that:

(86) $\lim_{\dot{e}(u) \to 0} \partial\mathbf{D}(e,\dot{e})/\partial\dot{e} = \lim_{\dot{e}(u) \to 0} \partial^2\mathbf{D}(e,\dot{e})/\partial\dot{e}^2 = \lim_{\dot{e}(u) \to 0} \partial^3\mathbf{D}(e,\dot{e})/\partial\dot{e}^3 = 0.$

The above nullifications are essential conditions (among the others, of course!) to apply the L'Hospital's rule several times by turns. Finally, taking the limit $\dot{e}(u) \to 0$ after the fourth differentiation:

(87) $\lim_{\dot{e}(u) \to 0} \partial^4\mathbf{D}(e,\dot{e})/\partial\dot{e}^4 = 6(8 + 24e^2 + 3e^4)(1 - e^2)^{-1/2},$

we obtain a non-zero result. In view of the fact that the fourth derivative $\partial^4(\dot{e}^4)/\partial\dot{e}^4 = 24 \neq 0$, and taking into account the temporally disregarded factor (84), we arrive to the following conclusion:

(88) $\lim_{\dot{e}(u) \to 0}\mathbf{H_4}(e,\dot{e}) = \lim_{\dot{e}(u) \to 0}\{\pi\mathbf{D}(e,\dot{e})\dot{e}^{-4}(1 - e^2)^{-7/2}[1 - (e - \dot{e})^2]^{-1/2}\} = (6\pi/24)(8 + 24e^2 + 3e^4)(1 - e^2)^{-7/2}(1 - e^2)^{-1/2}(1 - e^2)^{-1/2} = (\pi/4)(8 + 24e^2 + 3e^4)(1 - e^2)^{-9/2}.$

A *direct* computation, based on the substitution $\dot{e}(u) = 0$ into the definition (79), leads to an equivalent to (88) final expression. That is to say, the evaluation procedure does not include in the intermediate calculations terms, containing the factor $\dot{e}(u)$ into their denominators. And, consequently, they do not suffer from a "peculiar" behaviour, when $\dot{e}(u)$ approaches zero. The integral $\mathbf{H_4}(e,\dot{e})$ becomes for $\dot{e}(u) = 0$ an already known function of $e(u)$:

$$(89)\qquad \mathbf{H_4}(e,0) \equiv \int_0^{2\pi}(1 + e\cos\varphi)^{-5}\, d\varphi \equiv \mathbf{A_5}(e,0) = (\pi/4)(\,8 + 24e^2 + 3e^4)(1 - e^2)^{-9/2},$$

where we have put into use the estimation (24). Like the previous considered cases, the conclusion which follows, implies that under the limit transition $\dot{e}(u) \to 0$, the solution (79) preserves its meaning and passes through the "divergence" point $\dot{e}(u) = 0$ in a continuous manner. Evidently, when both $e(u)$ and $\dot{e}(u)$ vanish simultaneously, the already derived expressions give the "right" answer $\mathbf{H_4}(0,0) = 2\pi$, in spite of the order by which $e(u)$ and $\dot{e}(u)$ attain their zero values.

## 3. Conclusions

The basic goal, which we intend to do in this paper, is to compute, by an analytical way, expressions for certain type integrals. They will be used in the forthcoming investigations of the dynamical equation of the elliptical accretion discs ([1], [2]). More precisely speaking, the integrands of these functions of the eccentricity $e(u)$ of the particle orbits, and their derivatives $\dot{e}(u) \equiv \partial e(u)/\partial u$. They contain into their denominators factors (or products of them) of the type $(1 + e\cos\varphi)^{\mathrm{i}}$ or $[1 + (e - \dot{e})\cos\varphi]^{\mathrm{j}}$. The powers **i** and **j** may take integer values 1, 2, 3, 4, 5 and so on. We stress that the considered integrals do not include into their nominators factors other than unity. Therefore, the integers i and j are always positive. Of course, we have limited us to a minimum set of numbers of these powers - such, which will be enough, in view of the future applications of the analytical solutions for these integrals. Although there is not (at least an obvious) doubt, that the used in the present paper (essentially recurrent) approach for analytical evaluations of the integrals $\mathbf{A_i}(e,\dot{e})$, $\mathbf{J_i}(e,\dot{e})$ and $\mathbf{H_i}(e,\dot{e})$, ($\mathbf{i}$ = 1, 2, 3,…), may be extended for arbitrary integers **i**, we do not solve this general problem. That is to say, we do not try to obtain any common expressions for each of these functions of $e(u)$ and $\dot{e}(u)$, valid for *arbitrary* powers $\mathbf{i}$ = 1, 2, 3,…. This would be an extended mathematical task, which would be beyond the scope of our efforts to analyze a concrete physical problem – the dynamical equation of the *stationary* elliptical accretion discs.

An essential peculiarity, which occurs for almost all the calculations of the above mentioned integrals, is that in the intermediate results appear terms, which may be divergent for some values of $e(u)$, $\dot{e}(u)$ or the difference $e(u) - \dot{e}(u)$. In the final expressions, representing the final solutions of the integrals, these peculiarities may present ***or*** not present. These indeterminacies can be overcome by means of *direct* substitutions of the above noted "peculiar" meanings into the initial definitions of $\mathbf{A_i}(e,\dot{e})$, $\mathbf{J_i}(e,\dot{e})$ and $\mathbf{H_i}(e,\dot{e})$, ($\mathbf{i}$ = 1, 2, 3,…), and then performing the integration. As a rule, this procedure is more easily fulfilled, and, fortunately, does not involve, at any stage of the calculations, the considered type of peculiarity. As it becomes evident, after the comparison of the two solutions, we strike with the following two situations:

(*i*) The final result for the expression, obtained through the "peculiar" intermediate terms, does not involve similar "peculiar" terms. Consequently, it is "regular" with respect to the substitution into it of the "singular" $e(u)$ and $\dot{e}(u)$. It is remarkable that the two ways (with singular intermediate terms and *direct* computation, without passing through such singular terms) give identical expressions. This is, of course, a favorable property, because there is not a necessity to point out the method, by which the considered formula is obtained.

(*ii*) The final result for the expression, obtained through the "peculiar" intermediate terms, retains its indeterminacy for the considered (caused by the divergent intermediate terms), "peculiar" values of $e(u)$ or/and $\dot{e}(u)$. Then, it turns out that it is possible, by the use of the L'Hospital's rule, to reveal these indeterminacies of the type 0/0. Again, it is worthy to note that the evaluated in this way (by means of the limit transitions $e(u) \to 0$, or/and $\dot{e}(u) \to 0$) expressions coincide with those, computed through the *direct* substitution of the "problem" values of $e(u)$ and $\dot{e}(u)$ into the integrals, which we want to evaluate analytically. Such a continuous transition enables us to use into the applications the derived formulas, without to specify the path by which they are established. And also **not to worry** whether the applications of the above computed analytical expressions for the integrals $\mathbf{A_i}(e,\dot{e})$, ($\mathbf{i}$ = 1, …, 5), $\mathbf{J_k}(e,\dot{e})$ and $\mathbf{H_k}(e,\dot{e})$, ($\mathbf{k}$ = 1, …, 4) do introduce any type their own divergence into the evaluated (more complex) composite expressions, of which they are parts.

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# АНАЛИТИЧНО ПРЕСМЯТАНЕ НА ДВА ИНТЕГРАЛА, ВЪЗНИКВАЩИ В ТЕОРИЯТА НА ЕЛИПТИЧНИТЕ АКРЕЦИОННИ ДИСКОВЕ. I. РЕШАВАНЕ НА СПОМАГАТЕЛНИТЕ ИНТЕГРАЛИ, ПОЯВЯВАЩИ СЕ ПРИ ТЯХНОТО ИЗЧИСЛЯВАНЕ

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## Резюме

Настоящата работа е част от едно обширно аналитично изследване на динамичното уравнение, определящо пространствената структура на *стационарните* елиптични акреционни дискове, съгласно модела на Любарски и др. [1]. При математическото описание на задачата са използвани като параметри ексценрицитета $\boldsymbol{e(u)}$ на орбитите на частиците и неговата производна $\boldsymbol{\dot{e}(u) \equiv de(u)/du}$, където

$\boldsymbol{u \equiv ln(p)}$, $\boldsymbol{p}$ е фокалният параметър на разглежданата орбита. В течение на процеса на опростяване на това уравнение, възниква необходимостта от

аналитично оценяване на интеграли от следните типове: $\mathbf{A_i}(\boldsymbol{e},\dot{\boldsymbol{e}}) = \int_0^{2\pi}(1 + e\cos\varphi)^{-i}\, d\varphi$,

($\mathbf{i}$ = 1, …, 5), $\mathbf{J_k}(\boldsymbol{e},\dot{\boldsymbol{e}}) = \int_0^{2\pi}(1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-k}\, d\varphi$ и $\mathbf{H_k}(\boldsymbol{e},\dot{\boldsymbol{e}}) = \int_0^{2\pi}(1 + e\cos\varphi)^{-k}[1 + (e - \dot{e})\cos\varphi]^{-1}\, d\varphi$, ($\mathbf{k}$ = 1, …, 4). В тези формули $\boldsymbol{\varphi}$ е азимуталният ъгъл, върху който е извършено усредняването. Подходът при решаването на задачата е, фактически, рекурсивен. Най-напред ние оценяваме интегралите за най-малките стойности на **i** и **k** (т.е., **i** и **k** равни на единица). След това ние преминаваме към следващите стъпки, постепенно увеличавайки целочислените степени **i** или **k**, докато се достигнат указаните стойности 5 или 4, съответно. Специално внимание е посветено на тези значеия на $\boldsymbol{e(u)}$ и $\boldsymbol{\dot{e}(u)}$ (и тяхната разлика $\boldsymbol{e(u) - \dot{e}(u)}$), които евентуално могат да причинят разходимости в промеждутъчните или крайните изрази. Показано е, въпреки възникването на такива затруднения, че те могат да бъдат преодоляни посредством прякото заместване на “особените” стойности на $\boldsymbol{e(u)}$ и/или $\boldsymbol{\dot{e}(u)}$ в интегралите**,** като чак след това се извършват изчисленията. Даже ако в знаменателите на крайните резутати се появяват множители равни на нула (в следствие на анулиранията на $\boldsymbol{e(u)}$, $\boldsymbol{\dot{e}(u)}$ или $\boldsymbol{e(u) - \dot{e}(u)}$), изразите не са разходящи, както ние сме доказали, използвайки правилото на Льопитал за разрешаване на неопределености от вида 0/0. Всички аналитични оценки на горенаписаните интеграли са извършени при ограниченията $\boldsymbol{|e(u) < 1|}$, $\boldsymbol{|\dot{e}(u) < 1|}$ и $\boldsymbol{|e(u) - \dot{e}(u)| < 1}$. Те са наложени поради физически съображения, с оглед на прилагането на тези решения във възприетата теория на елиптичните акреционни дискове.