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ESTIMATIONS OF TOTAL MASS AND DENSITY OF THE OBSERVABLE UNIVERSE BY DIMENSIONAL ANALYSIS

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The dimensional analysis has been carried out by means of three fundamental constants - the speed of the light in vacuum (c), the universal gravitational constant (G) and the Hubble constant (H). The mass dimension quantity $m \sim c^3/(GH)$ ~ 10^{53} *kg, derived by this approach, practically coincides with Hoyle-Carvalho formula for the mass of the observable universe, obtained by a totally different approach. It has been shown that this value is close to the mass of the Hubble sphere. Besides, by dimensional analysis it has been found that the total density of the universe, including the dark matter and the dark energy, is of the order of a magnitude of the critical density of the universe* $\rho \sim H^2/G \sim \rho_c \approx 10^{-26}$ *kg m*$^{-3}$.

Key words: dimensional analysis; fundamental constants; mass of the universe; total density of the universe

1. Introduction

The observable universe consists of the galaxies and other matter that we can in principle observe from Earth in the present day, because light (or other signals) from those objects has had time to reach us since the beginning of the cosmological expansion. Because the universe is homogeneous and isotropic in large scale, the distance to the edge of the observable universe is roughly the same in every direction. Therefore, the observable universe is a three dimensional sphere centered on the observer.

Until recently scarce information has been available concerning the mass, density and geometry of the universe. Recent observations have shown that the average density of the bright matter (stars, galaxies, quasars, etc.) is less than 1 % of the critical density of the universe ρ_c [1].

(1) $$\rho_c = \frac{3H^2}{8\pi G} \approx 10^{-26}\ kg\ m^{-3}$$

On other side, the rotational curves of spiral galaxies [2] and the stability of rich clusters of galaxies [3] infer total density of the matter $\Omega_M = \rho_M / \rho_c \sim 0.25$, in units of the critical density ρ_c. According to the Big Bang Nucleosynthesis, the density of the ordinary (baryonic) matter is $\Omega_B < 0.05$ [4], whereas the density of the cold dark matter $\Omega_C > 0.20$ [5].

The most trustworthy total matter density $\Omega_0 = \bar{\rho} / \rho_c$ has been determined by measurements of the dependence of the anisotropy of the Cosmic Microwave Background (*CMB*) upon the angular scale. The recent observations show that $\Omega_0 \approx 1 \pm \Delta\Omega$, where the error $\Delta\Omega = 0.02$ [6, 7], i.e. the density of the universe is close to the critical one and the universe is asymptotically flat (Euclidean). Recent distant SNeIa observations have shown accelerating expansion of the universe produced by the dark energy, possessing density $\Omega_\Lambda > 0.70$ [8 – 11]. Therefore, the bulk of density of the universe consists from dark energy and cold dark matter $\Omega_0 = \Omega_B + \Omega_C + \Omega_\Lambda \approx 1$.

In this paper we have shown that the mass and density of the observable universe can be estimated by dimensional analysis using the fundamental constants – the speed of the light in vacuum (*c*), the gravitational constant (*G*) and the Hubble constant (*H*), without any information for the total density of the universe.

The dimensional analysis is a conceptual tool often applied in physics to understand physical situations involving certain physical quantities [12 – 14]. It is routinely used to check the plausibility of the derived equations and computations. When it is known, the certain quantity with which other determinative quantities would be connected, but the form of this connection is unknown, a dimensional equation is composed for its finding. In the left side of the equation, the unit of this quantity q_0 with its dimensional exponent has been placed. In the right side of the equation, the product of units of the determinative quantities q_i rise to the unknown exponents n_i, has been placed $[q_0] \sim \prod_{i=1}^{n} [q_i]^{n_i}$, where n is positive integer and the exponents n_i are rational numbers. Most often, the dimensional analysis is applied in mechanics and other fields of modern physics, where there are many problems with a few determinative quantities. Many interesting and important problems related to the fundamental constants have been considered in [15 – 17].

The discovery of the linear relationship between recessional velocity of distant galaxies, and distance $v = Hr$ [18] introduces new fundamental constant in physics and cosmology – the famous Hubble constant (H). Hubble constant determines the age of the universe H^{-1}, the Hubble distance cH^{-1}, the critical density of the universe ρ_c, and other large-scale properties of the universe. Because of the importance of the Hubble constant, in the present paper we include H in the dimensional analysis. Thus, the Hubble constant will represent the cosmological phenomena in new derived fundamental mass. According to the recent cosmology, the Hubble 'constant' slowly decreases with the age of the universe, but there are indications that other constants, especially gravitational and fine structure constants also vary with time [19, 20]. That is why, the Hubble constant could deserve being treated on an equal level with the other constants.

2. Estimation of total mass of the observable universe by dimensional analysis

The Plank mass $m_P \sim \sqrt{\frac{\hbar c}{G}}$ has been derived from Planck [21] by dimensional analysis using the fundamental constants – c, G and the reduced Plank constant ($\hbar$). Since the constants c, G and $\hbar$ represent three very basic aspects of the universe (i.e. the relativistic, gravitational and quantum phenomena), the Plank mass appears to a certain degree a unification of these phenomena. The Plank mass have many important aspects in modern physics. One of them is that the energy equivalent of

Planck mass $E_P = m_P c^2 \sim \sqrt{\frac{\hbar c^5}{G}} \sim 10^{19}\ GeV$ appears as unification energy of the fundamental interactions [22].

Analogously, we seek a mass dimension quantity *m* composed from the fundamental constants – the speed of the light (*c*), the gravitational constant (*G*) and the Hubble constant (*H*), using dimensional analysis. Here we replace Planck constant with Hubble constant assuming that the quantum phenomena described by $\hbar$ are negligible in relation to the cosmological phenomena described by *H*. Consequentially, so we can write equation (2):

(2) $$m = kc^{n_1} G^{n_2} H^{n_3},$$

where n_1, n_2 and n_3 are unknown exponents to be determined by matching the dimensions of both sides of the equation and *k* is dimensionless parameter of the order of magnitude of unit. Using the symbol *L* for length, *T* for time, *M* for mass, and writing "[*x*]" for the dimensions of some physical quantity *x*, we have the following:

(3) $$[c] = LT^{-1}$$
$$[G] = M^{-1}L^3T^{-2}$$
$$[H] = T^{-1}$$

The dimensions of the left and right sides of the equation (2) must be equal. Therefore:

(4) $$[m] = [c]^{n_1}[G]^{n_2}[H]^{n_3}$$

Taking into account the dimensions of quantities in formula (4) we obtain:

(5) $$L^0T^0M^1 = (LT^{-1})^{n_1}(L^3T^{-2}M^{-1})^{n_2}(T^{-1})^{n_3} = L^{n_1+3n_2}T^{-n_1-2n_2-n_3}M^{-n_2}$$

We find the system of linear equations from (5):

(6) $$n_1 + 3n_2 = 0$$
$$-n_1 - 2n_2 - n_3 = 0$$
$$-n_2 = 1$$

The determinant Δ of the system is:

(7) $$\Delta = \begin{vmatrix} 1 & 3 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & 0 \end{vmatrix} = -1$$

The determinant $\Delta \neq 0$. Therefore, the system has an unique solution. We find this solution by Kramer's formulae (8):

(8) $$n_1 = \frac{\Delta_1}{\Delta}, \quad n_2 = \frac{\Delta_2}{\Delta}, \quad n_3 = \frac{\Delta_3}{\Delta}$$

where $\Delta_1 = -3$, $\Delta_2 = 1$ and $\Delta_3 = 1$.

Therefore, the exponents $n_1 = 3, n_2 = -1, n_3 = -1$. Replacing obtained values of exponents in equation (2) we find formula (9) for the mass m_1:

(9) $$m \sim \frac{c^3}{GH}$$

First of all, the formula (9) has been derived by dimensional analysis in [23]. This formula is close to the Hoyle formula [24] for the mass of the universe $M = c^3/(2GH)$ and perfectly coincides with Carvalho formula [25] for the mass of the universe $M \sim c^3/(GH)$, obtained by totally different approaches, Steady State Theory [26] and Large Number Hypothesis, respectively [19].

The strict linearity of the Hubble law has been confirmed in [27] by SNeIa observations. The Hubble sphere is the sphere where the recessional velocity of the galaxies is equal to the speed of the light in vacuum *c*, and according to the Hubble law $v = c$ when $r = cH^{-1}$. Thus, the Hubble sphere appears a three-dimensional sphere, centered on the observer, having radius $r = cH^{-1}$ and density $\overline{\rho} \approx \rho_c$. Therefore, the mass of the Hubble sphere is:

$$(10)\quad M_H \approx \frac{4}{3}\pi r^3 \rho_c = \frac{4}{3}\pi \frac{c^3}{H^3}\frac{3H^2}{8\pi G} = \frac{c^3}{2GH}$$

Obviously, the mass of the Hubble sphere (10) coincides with the Hoyle formula for the mass of the universe and differs from formula (9) with dimensionless parameter $k = ½ \sim 1$.

The recent experimental values of c, G and H are used – c = 299 792 458 $m\ s^{-1}$, G = 6.673×10^{-11} $m^3\ kg^{-1}\ s^{-2}$ [28] and $H \approx 70$ $km\ s^{-1}\ Mps^{-1}$ [29]. Replacing this values in (9) we obtain $m \sim 1.76\times10^{53}$ kg. Therefore, the enormous mass $m \sim c^3/(GH)$ would be identified with mass of the observable universe. Strictly speaking, the size of the observable universe determines from the cosmological horizon and depends from cosmological model, yet the former roughly coincides with Hubble distance.

3. Estimation of total density of the observable universe by dimensional analysis

Analogously, a quantity ρ having dimension of density could be composed by means of the fundamental constants c, G and H:

$$(11)\quad \rho = kc^{n_1}G^{n_2}H^{n_3},$$

where k is a dimensionless parameter of the order of magnitude of unit.

By dimensional analysis, we find the system of linear equations:

$$(12)\quad \begin{aligned} &n_1 + 3n_2 = -3 \\ &-n_1 - 2n_2 - n_3 = 0 \\ &-n_2 = 1 \end{aligned}$$

The determinant of the system is $\Delta = -1 \neq 0$. Therefore, the system has a unique solution, which we obtain by the Kramer's formulae again:

$$(13)\quad \begin{aligned} n_1 &= \frac{\Delta_1}{\Delta} = 0 \\ n_2 &= \frac{\Delta_2}{\Delta} = -1 \\ n_3 &= \frac{\Delta_3}{\Delta} = 2 \end{aligned}$$

Replacing obtained values of the exponents in equation (11) we find formula (14) for the density ρ:

$$\rho \sim \frac{H^2}{G} \approx 7.93 \times 10^{-26}\ kg\ m^{-3} \tag{14}$$

As mentioned in Section 1, the recent Cosmic Microwave Background (*CMB*) observations have shown that the total density of the universe $\bar{\rho}$ is close to the critical density ρ_c:

$$\bar{\rho} = \Omega\rho_c \approx \rho_c = \frac{3H^2}{8\pi G} \approx 10^{-26}\ kg\ m^{-3} \tag{15}$$

It deserves to note that the formula (14) correctly determines the dependence of the total density of the universe on the Hubble and gravitational constants. Besides, the density ρ derived by means of the fundamental constants *c*, *G* and *H* coincides with formula (15) for the total density of the universe with accuracy to a dimensionless parameter $k = 3/(8\pi)$ of the order of a magnitude of a unit.

The formula (14) could be derived by means of other set of fundamental constants, namely ($\hbar$, *G*, *H*). Actually, a quantity ρ having dimension of density could be composed by means of the fundamental constants $\hbar$, *G* and *H*:

$$\rho = k\hbar^{n_1} G^{n_2} H^{n_3}, \tag{16}$$

where *k* is a dimensionless parameter of the order of magnitude of unit.

By dimensional analysis, we obtain the respective system of linear equations:

$$\begin{aligned} 2n_1 + 3n_2 &= -3 \\ -n_1 - 2n_2 - n_3 &= 0 \\ n_1 - n_2 &= 1 \end{aligned} \tag{17}$$

The determinant of the system $\Delta = -5 \neq 0$ and the system have a solution coinciding with the solution of the system (12):

$$n_1 = \frac{\Delta_1}{\Delta} = 0$$

(18) $$n_2 = \frac{\Delta_2}{\Delta} = -1$$

$$n_3 = \frac{\Delta_3}{\Delta} = 2$$

Replacing the exponents in (16) we again obtain equation (14) for the total density of the universe. Therefore, the mass dimension quantity composed by means of the set of fundamental constants ($\hbar$, G, H) coincides with equation (14) found by means of the set of constants (c, G, H). Thus, the dimensional analysis automatically rejects inappropriate determinative quantities from the equations, such as c and $\hbar$ in the above examined cases.

4. Conclusions

The dimensional analysis has been carried out by means of three fundamental constants - the speed of the light in vacuum (c), the universal gravitational constant (G) and the Hubble constant (H). The mass dimension quantity $m \sim c^3/(GH)$ ~ 10^{53} *kg*, derived by this approach, practically coincides with Hoyle-Carvalho formula for the mass of the observable universe, obtained by a totally different approach. It has been shown that this value is close to the mass of the Hubble sphere. Besides, by dimensional analysis it has been found that the total density of the universe, including the dark matter and the dark energy, is of the order of a magnitude of the critical density of the universe $\rho \sim H^2/G \sim \rho_c \approx 10^{-26}$ $kg\ m^{-3}$. It deserves to note that these formulae have been derived without consideration of any cosmological model and the formula for the total mass of the observable universe (9) has been obtained by means of the fundamental parameters c, G and H only, with no information for the total density of the universe.

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ОЦЕНКИ НА ТОТАЛНАТА МАСА И ПЛЪТНОСТ НА НАБЛЮДАЕМАТА ВСЕЛЕНА ПОСРЕДСТВОМ АНАЛИЗ НА РАЗМЕРНОСТИТЕ

Д. Вълев

Резюме

Проведен бе анализ на размерностите с помощта на три фундаментални константи, а именно – скоростта на светлината във вакуум (c), универсалната гравитационна константа (G) и константата на Хъбъл (H). Величината с размерност на маса $m \sim c^3/(GH) \sim 10^{53}$ *кg*, изведена посредством този метод практически съвпада с формулата на Хойл-Карвальо за масата на наблюдаемата Вселена, получена посредством съвършенно различен метод. Показано бе че, тази стойност е близка до масата на сферата на Хъбъл. Освен това, посредством анализ на размерностите бе установено, че тоталната плътност на Вселената, включително тъмната материя и тъмната енергия, е от порядъка на критичната плътност на Вселената $\rho \sim H^2/G \sim \rho_c \approx 10^{-26}$ *кg m*$^{-3}$.