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THIN VISCOUS ELLIPTICAL ACCRETION DISCS WITH ORBITS SHARING A COMMON LONGITUDE OF PERIASTRON. VI. SIMPLIFICATION OF THE DYNAMICAL EQUATION

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Abstract

We continue the series of papers, devoted to the investigation and simplification of the dynamical equation, governing the structure of the **stationary** *elliptical accretion discs. These studies are in the frameworks, specified according to the model of Lyubarskij et al. [7]. In addition to the previous examinations, we find one more linear relation between the coefficients of this second order ordinary differential equation, which enables us to eliminate effectively at least four of them. This is in the course of our approach to reduce the number of these functions, depending on the eccentricity, its derivative and the power* $\boldsymbol{n}$ *in the viscosity law* $\boldsymbol{\eta = \beta \Sigma^{n}}$*. They appear in the equation during the process of averaging (i.e. integrating) over the azimuthal angle of the elliptical orbits. At the present stage of the investigations, there still remain three integrals of the indicated type. Except the case of integer values of* $\boldsymbol{n}$*, their analytical solutions are not known. In connection with the linear dependence or independence of these functions (this is a subject of our future studies), the dynamical equation of the elliptical accretion discs may be split into a system of corresponding number of more simple equations about the unknown eccentricities of the particle orbits. Such an approach is in accordance with our base line, carried out through the referred series of papers, to make a progress, as much as possible, into the solving of the task by means of* **purely analytical** *methods. And only when the further advance in this way (if the final solution is not already attained) is so complicated, that it is impasse, to use numerical simulations.*

1. Introduction

There are numerous observational examples of accretion discs, orbiting around one of the components in a binary stellar system. The observational evidences, proving the existence of such astronomical objects, are based mainly on the accretion processes on to the compact object, located at the center of the disc. It may happen that the supply of mass to the central object can vary for different physical and geometrical reasons. This is possible to occur during the state transitions of the accretion flow or due to the variable interactions in the eccentric binary system. Such an activity, in principle, is a subject of damping by the viscosity of the disc matter. The mass supply on to the primary star (black hole, neutron star, white dwarf) is, in the end, determined by the mass supply at the outer disc edge. The authors of the investigation [1] compare this physical process for two accretion disc models: such as with finite and with infinite sizes. They find significant differences between these two cases. Namely, the infinite disc solution overestimates the viscous damping. They conclude that, generally speaking, the damping becomes very strong when the viscous time at the outer edge of the disc turns longer than the modulation time scale [1]. We consider the above example, in order to underline how important may be the right evaluation even of a *single* parameter, when we describe the accretion flow events.

The variations of the properties of the accretion discs are subject not only to *internal* changes of their parameters, but also as consequences of *external* influences on the stellar system. The interactions between a "star + accretion disc" system and another star will perturb the disc, possibly resulting in significant modifications of the disc structure and its physical properties. It is suggested that such encounters are capable to trigger fragmentation of the disc, to form brown dwarfs or gas giant planets [2]. In the later paper are simulated star-star encounters, where the primary star has a self-gravitating, marginally stable protostellar disc, and the secondary star has not disc. The results of this investigation of the variations of the disc structure and its dynamics may be summarized in the following way. The stellar encounter is to prohibit the fragmentation, because compressive and shock heating stabilizes the disc and the radiative cooling is insufficient to trigger the gravitational instability. The conclusions from these simulations [2] show, that the encounter strips the outer regions of the disc. This can be realized either by tidal tails or by a capture of matter to form a disc around the secondary star. As a final result, the interaction triggers a readjustment if the initially existing primary accretion disc turns to such with a steeper surface density profile. We conclude from such studies, that accretion discs have not only good possibilities to survive the encounters with the other stars, but also to preserve their relatively smooth spatial distribution of different physical parameters, characterizing the global accretion flow. It is worthy to note, that the *fractal* concepts have recently been introduced in the accretion disc theory as a new feature. As pointed out by Roy & Ray [3], due to the fractal nature of the flow, its continuity condition undergoes modifications. They show (through completely analytical solutions of the equilibrium point conditions), that their results give indications, that the *fractal properties enable the flow to behave like an effective continuum of lesser density*. The mass accretion rate exhibits a fractal scaling behavior, and the entire fractal accretion disc is stable under *linearized* dynamic perturbations.

As we know, the accretion discs in the binary stellar systems are frequently occurred objects and naturally arises the question about the relation of the shape of the disc and the eccentricity of the binary orbit. Marzari et al. [4] study the evolution of circumstellar massive discs around the primary star of a binary system. Especially, they concentrate on the computation of the disc eccentricity and its dependence on the binary orbit eccentricity. The conclusions are that the self-gravitation of the massive discs leads to discs that have (on average) low eccentricity. They establish that the orientation of the disc, computed with the standard dynamical method, always librates, instead of circulating as in the simulations *without* self-gravitation. The simulations show that the accretion disc eccentricity decreases with the binary eccentricity. This result is found also in models *without* self-gravitation. Generally speaking, the investigation [4] is in agreement with the statement that the disc self-gravitation appears to be an important factor in determining the evolution of the *massive* accretion discs in the binary systems. One additional complicating factor, which possibly affects the shape of the accretion flow, is its orientation with respect to the spin axis of the central body. Modelling of the overall shape of an accretion disc in a semidetached binary system is performed, for example, in the paper of Martin et al. [5]. In this investigation, the mass is transferred on to a spinning black hole, which spin axis is misaligned with the orbital rotation axis of the binary. It is assumed that the accretion disc around the black hole is in a *steady* state. It turns out, that its outer regions are subject to differential precession, caused by the tidal torques of the companion star.

The later tend to align the outer parts of the disc with the orbital plane. But the inner regions of the disc are subject to differential precession caused by the general relativity action (Lense-Thirring effect). Such an influence tends to align the rotation of the inner parts of the accretion disc with the spin of the black hole. There are many other examples, both theoretical and observational, illustrating that the disc midplane may be inclined relative to the binary orbital plane. Under suitable physical conditions, similar geometrical configuration is expected to induce warping and rigid body precession of the disc. Fragner & Nelson [6] find that thick discs (with aspect ratio = 0.5 and low viscosity parameter α) precess as rigid bodies with very little warping or twisting. They are observed to align with the binary orbital plane on the viscous evolution time. On the other hand, thinner discs with higher viscosity, in which warp communication is less efficient, develop significant twists, before to achieve a state of rigid body precession. Under the most extreme conditions considered in [6] (with aspect ratio = 0.01; $\alpha = 0.1$ and $\alpha = 0.005$), it is established that the accretion discs can become broken or disrupted by the strong differential precession. Discs that become highly twisted are observed to align with the binary orbital plane on time scales much shorter than the viscous time scale and, possibly, on the precession time [6]. These examples, concerning the complicated internal and external interactions in the accretion flows, demonstrate some of the difficulties, which may occur, when the shape of the disc must be established in a quantitative manner. In the present paper, we investigate a particular accretion disc model, having an elliptical shape. The trajectories of the disc particles are assumed *ad hoc* to be ellipses, sharing a common longitude of periastron. More precisely, the dynamical equation, with which we shall deal, describes and governs the structure of the disc in the model developed by Lyubarskij et al. [7], and which generalizes the standard α-disc model of Shakura & Sunyaev [8]. These two papers do not involve in their considerations strongly magnetized accretion flows. Taking into account such a circumstance, it is a matter of interest to mention the recent study of Murphy et al. [9], devoted to the *large-scale* magnetic fields in the viscous resistive accretion discs. According to the theory of the winds from cold steady-state discs, having near Keplerian velocity field, there is a necessity for a large-scale magnetic field at near equipartition strength to present. However, as mentioned in [9], this required minimum magnetization (for these disc models) has never been tested with time dependent simulations. In order to eliminate this omission, Murphy et al. [9] investigate the time evolution of a Shakura-Sunyaev accretion disc [8], threaded by a weak magnetic field. Its strength is such that the disc magnetization falls off rapidly with the radius. The results lead to the conclusion that the *large-scale* magnetic field introduces only a small perturbation to the disc structure and the accretion remains driven by the dominant viscous torque. However, their numerical simulations reveal that a superfast magnetosonic jet is observed to be launched from the innermost regions of the disc and *continues to be a stationary event for a long time*. The main conclusions, following from these numerical simulations, are that the astrophysical discs with superheated surface layers could drive analogous outflows, even if their midplane magnetization is low. In order the accretion process to proceed, the turbulent viscosity must extract a sufficient angular momentum. The authors of the investigation [9] conclude that the magnetized outflows are no more than byproduct, rather than a fundamental driver of the accretion. Nevertheless, if the midplane magnetization increases towards the center of the accretion disc, a natural transition to an inner *jet-dominated* disc could eventually be achieved. We shall pick out a little more attention to the important process of the angular momentum transfer in the accretion flows.

2. Mechanisms of angular momentum transport

It is believed that the microphysical viscosity is too small to produce the observed protoplanetary accretion disc lifetimes. Instead of that, it is suggested a new approach, based on the turbulent transport. In the later case, the turbulent motion takes the place of thermal motion. Though the source of such turbulence remains a matter of discussion, this process can provide the correct order of magnitude of the observed accretion rates in these objects for reasonable surface densities [10]. It may happen that the main accretion mechanism is not the turbulent viscosity, as can be seen in the situation with the magnetorotational instability. According to the numerical simulations performed in [10], the requirement for energy conservation is a significant constraint on the accretion driving processes, such as the magnetorotational instability. The mechanism of angular momentum transport in accretion discs is debated for a long time. In this stream of investigations, it should be noted that although the magnetorotational instability appears to be a promising explanation of the accretion events, in the poorly ionized regions of accretion discs there may not be favorable conditions for this instability to operate. In the research [11] is revisited the possibility of transporting angular momentum by thermal *convection*. There is shown that strongly turbulent convection can drive outward angular momentum transport at a rate that is (under certain conditions) compatible with the observed rate in the discs. The results of Lesur & Ogilvie [11] are indicative that *convection* might be another way to explain ***global*** disc evolution. Such a scenario will be realistic provided that a sufficiently unstable *vertical* temperature profile can be maintained.

Another recent research, devoted to the role of magnetic field in the angular momentum transport, is performed in the paper [12]. The physical situation (advection-dominated accretion flow with a toroidal magnetic field) and the geometric configuration (quasi-spherical accretion flow) are much different in comparison with the standard disc model of Shakura & Sunyaev [8], and also from the model Lyubarskij et al. [7]. Nevertheless, it is worthy to note, that the conclusions in [12] may have, to some extent, a reference to the former two models. In the research of Khesali & Faghei [12] it is assumed that (like in [7] and [8]), the angular momentum transport is due to the viscous turbulence and the α-prescription is used for the kinematic coefficient of viscosity. In this paper [12], a self-similar solution is used, in order to solve the equations that govern the dynamical behaviour of the accretion flow. According to the conclusions of Khesali & Faghei [12], their solution provides some insights into the dynamics of *quasi-spherical* accretion flows and avoids many of the strictures of *steady* self-similar solutions. The results in [12] show that the behaviour of the physical quantities in a ***dynamical*** advection-dominated accretion flow is different from that for a ***steady*** accretion flow or a disc using a polytropic approach. This model also implies that the flow has a differential rotation that is a sub-Keplerian at small radii and super-Keplerian at large radii. Such different results are also obtained if a polytropic accretion flow is used. Also, the behaviour of the advection-dominated flows in the presence of a large toroidal magnetic field implies that different results are obtained using *steady-state* self-similar models in contrast to the *dynamical* case. The above remarks have to be referred/related (in some conditional and restricted sense) to the ***steady-state*** case of the model of Lyubarskij et al. [7], which dynamical equation we are going to simplify further in the present paper. The same note is also significant in view of the fact that the classical models [7] and [8] do not involve considerations of large-scale magnetic fields. Restricting our attention only to the particular case of steady-state elliptical accretion discs with orbits sharing a common longitude of periastron, we shall preliminary eliminate from our treatment the more unusual so called "mini-discs". It is believed that they arise due to the accretion on to black holes in wind-fed binaries and collapsars. Formation of such small rotating discs is combined with some peculiar properties. They accrete on the free-fall time-scale, *without the help of the viscosity*, and, nevertheless, they can have a high radiative efficiency [13]. We shall not, of course, apply our results to these inviscid "mini-discs". In principle, in the nature may exist even more "exotic" accretion discs. As pointed out by Zhang et al. [14], the temperature of the hot accretion flows around black holes is sufficiently high for the ignition of nuclear reactions. In the usual studies of the hot accretion flows, the viscous dissipation is considered as the only heating mechanism. In the same time, the heating caused by nuclear reactions is not considered at all. The calculations of Zhang et al. [14] indicate that the energy generation rate of nuclear reactions is *at most* one per cent of the viscous heating. Consequently, they are rather not important and the dynamics of the accretion flow can be calculated in the usual way, without the need to consider the heating due to the nuclear reactions.

3. Definitions and notations

With a view to be more explicit in our further exposition, we shall rewrite briefly some of the definitions and notations introduced and used in our earlier papers, dealing with the same problem. For more detailed descriptions and comments on this theme, the reader is directed to the paper ([15], section 2) and the references therein. We introduce the independent variable $u \equiv \ln p$, where p is the focal parameter of the ellipse, approximating the orbit of the considered disc particle. The eccentricity of the ellipse is denoted by $e \equiv e(u)$, understanding that the orbits of the particles, belonging to different regions of the accretion disc, may, generally speaking, have different shapes/elongations. Further we use the notation $\dot{e} \equiv \dot{e}(u) \equiv de(u)/du \equiv de/d(\ln p)$ for the ordinary derivative of the eccentricity $e(u)$. We shall consider the viscosity law $\eta = \beta\, \Sigma^{\,n}$, with η – viscosity parameter, β is a constant, Σ is the surface density of the accretion disc. The power n is assumed to be a constant for every examined case. In paper [15] are established several linear relations between the following integrals, obtained after the averaging over the azimuthal angle φ:

(1) $\mathbf{I_{0-}}(e,\dot{e},n) \equiv \int_0^{2\pi} (1 + e\cos\varphi)^{n-3}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi$,

(2) $\mathbf{I_{0+}}(e,\dot{e},n) \equiv \int_0^{2\pi} (1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+2)}\, d\varphi$,

(3) $\mathbf{I_j}(e,\dot{e},n) \equiv \int_0^{2\pi} (\cos\varphi)^{j}(1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi$; $\mathbf{j} = 0$, 1, 2, 3, 4.

In the earlier paper [15], we have shown how the integrals $\mathbf{I_4}(e,\dot{e},n)$, $\mathbf{I_2}(e,\dot{e},n)$ and $\mathbf{I_1}(e,\dot{e},n)$ can be expressed through linear relations of the integrals $\mathbf{I_0}(e,\dot{e},n)$, $\mathbf{I_{0-}}(e,\dot{e},n)$, and $\mathbf{I_{0+}}(e,\dot{e},n)$. Our present aim is to extend this approach, finding a linear relation between the last three integrals, namely, to determine how $\mathbf{I_0}(e,\dot{e},n)$ may be written out as a linear combination of $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$.

4. Elimination of the integral $\mathbf{I_0}(e,\dot{e},n)$

We have already obtained the following relation between the integrals $\mathbf{I_2}(e,\dot{e},n)$, $\mathbf{I_0}(e,\dot{e},n)$ and $\mathbf{I_1}(e,\dot{e},n)$ ([15], formula (57)):

(4) $[e + (n - 1)\dot{e}]\mathbf{I_2}(e,\dot{e},n) = (- e + n\dot{e})\mathbf{I_0}(e,\dot{e},n) - [1 + e(e - \dot{e})]\mathbf{I_1}(e,\dot{e},n)$.

To resolve the present problem, we begin with an another derivation of the integral $\mathbf{I_2}(e,\dot{e},n)$. As before, we suppose that $ne(u)[e(u) - \dot{e}(u)] \neq 0$ for the considered value of $u \equiv \ln p$. The particular cases, when this condition is violated, will be considered separately. According to definition (3):

(5) $\mathbf{I_2}(e,\dot{e},n) = \int_0^{2\pi} \cos^2\varphi(1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi =$

$= \int_0^{2\pi} (1 - \sin^2\varphi)(1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi =$

$= \int_0^{2\pi} (1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi\ +$

$+ (e - \dot{e})^{-1} \int_0^{2\pi} \sin\varphi(1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d[1 + (e - \dot{e})\cos\varphi] =$

$$= \mathbf{I_0}(e,\dot{e},n) - [n(e-\dot{e})]^{-1}\int_0^{2\pi} sin\varphi(1+ecos\varphi)^{n-2} d\{[1+(e-\dot{e})cos\varphi]^{-n}\} = \mathbf{I_0}(e,\dot{e},n) -$$

$$- [n(e-\dot{e})]^{-1}\{ sin\varphi(1+ecos\varphi)^{n-2}[1+(e-\dot{e})cos\varphi]^{-n}\Big|_0^{2\pi} -$$

$$- \int_0^{2\pi} cos\varphi(1+ecos\varphi)^{n-2}[1+(e-\dot{e})cos\varphi]^{-n} d\varphi +$$

$$+ (n-2)e\int_0^{2\pi} sin^2\varphi(1+ecos\varphi)^{n-3}[1+(e-\dot{e})cos\varphi]^{-n}\, d\varphi \} = \mathbf{I_0}(e,\dot{e},n) +$$

$$+ [n(e-\dot{e})]^{-1}\int_0^{2\pi}[1+(e-\dot{e})cos\varphi]\, cos\varphi(1+ecos\varphi)^{n-2}[1+(e-\dot{e})cos\varphi]^{-(n+1)} d\varphi -$$

$$- (n-2)e[n(e-\dot{e})]^{-1}\int_0^{2\pi}[1+(e-\dot{e})cos\varphi](1-cos^2\varphi)(1+ecos\varphi)^{n-3} \times$$

$$\times [1+(e-\dot{e})cos\varphi]^{-(n+1)} d\varphi = \mathbf{I_0}(e,\dot{e},n) + [n(e-\dot{e})]^{-1}\mathbf{I_1}(e,\dot{e},n) + n^{-1}\mathbf{I_2}(e,\dot{e},n) -$$

$$- (n-2)e[n(e-\dot{e})]^{-1}\int_0^{2\pi}(1-cos^2\varphi)(1+ecos\varphi)^{n-3}[1+(e-\dot{e})cos\varphi]^{-(n+1)} d\varphi -$$

$$- (n-2)en^{-1}\int_0^{2\pi}(1-cos^2\varphi)cos\varphi(1+ecos\varphi)^{n-3}[1+(e-\dot{e})cos\varphi]^{-(n+1)} d\varphi =$$

$$= \mathbf{I_0}(e,\dot{e},n) + [n(e-\dot{e})]^{-1}\mathbf{I_1}(e,\dot{e},n) + n^{-1}\mathbf{I_2}(e,\dot{e},n) -$$

$$- (n-2)e[n(e-\dot{e})]^{-1}\int_0^{2\pi}(1+ecos\varphi)^{n-3}[1+(e-\dot{e})cos\varphi]^{-(n+1)} d\varphi +$$

$$+ (n-2)e[n(e-\dot{e})]^{-1}\int_0^{2\pi} cos^2\varphi(1+ecos\varphi)^{n-3}[1+(e-\dot{e})cos\varphi]^{-(n+1)} d\varphi -$$

$$- (n-2)en^{-1}\int_0^{2\pi} cos\varphi(1+ecos\varphi)^{n-3}[1+(e-\dot{e})cos\varphi]^{-(n+1)} d\varphi +$$

$$+ (n-2)en^{-1}\int_0^{2\pi} cos^3\varphi(1+ecos\varphi)^{n-3}[1+(e-\dot{e})cos\varphi]^{-(n+1)} d\varphi =$$

$$= \mathbf{I_0}(e,\dot{e},n) + [n(e-\dot{e})]^{-1}\mathbf{I_1}(e,\dot{e},n) + n^{-1}\mathbf{I_2}(e,\dot{e},n) - (n-2)e[n(e-\dot{e})]^{-1}\mathbf{I_{0\text{-}}}(e,\dot{e},n) +$$
$$+ (n-2)e[n(e-\dot{e})]^{-1}[e^{-1}\mathbf{I_1}(e,\dot{e},n) - e^{-2}\,\mathbf{I_0}(e,\dot{e},n) + e^{-2}\,\mathbf{I_{0\text{-}}}(e,\dot{e},n)] -$$
$$- (n-2)en^{-1}[e^{-1}\mathbf{I_0}(e,\dot{e},n) - e^{-1}\mathbf{I_{0\text{-}}}(e,\dot{e},n)] +$$
$$+ (n-2)en^{-1}[e^{-1}\mathbf{I_2}(e,\dot{e},n) - e^{-2}\,\mathbf{I_1}(e,\dot{e},n) + e^{-3}\,\mathbf{I_0}(e,\dot{e},n) - e^{-3}\,\mathbf{I_{0\text{-}}}(e,\dot{e},n)]\,.$$

Computing of the last three integrals in the right-hand-side of the above equation is expressed in the next three equalities:

(6) $$\int_0^{2\pi} cos\varphi(1 + ecos\varphi)^{n-3}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}d\varphi =$$

$$= e^{-1}\int_0^{2\pi}(1 + ecos\varphi - 1)(1 + ecos\varphi)^{n-3}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}d\varphi =$$

$$= e^{-1}\mathbf{I_0}(e,\dot{e},n) - e^{-1}\mathbf{I_{0\text{-}}}(e,\dot{e},n)\ ,$$

(7) $$\int_0^{2\pi} cos^2\varphi(1 + ecos\varphi)^{n-3}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}d\varphi =$$

$$= e^{-1}\int_0^{2\pi} cos\varphi(1 + ecos\varphi - 1)(1 + ecos\varphi)^{n-3}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}d\varphi =$$

$$= e^{-1}\mathbf{I_1}(e,\dot{e},n) - e^{-2}\,\mathbf{I_0}(e,\dot{e},n) + e^{-2}\,\mathbf{I_{0\text{-}}}(e,\dot{e},n)\ ,$$

(8) $$\int_0^{2\pi} cos^3\varphi(1 + ecos\varphi)^{n-3}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}d\varphi =$$

$$= e^{-1}\int_0^{2\pi} cos^2\varphi(1 + ecos\varphi - 1)(1 + ecos\varphi)^{n-3}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}d\varphi =$$

$$= e^{-1}\mathbf{I_2}(e,\dot{e},n) - e^{-2}\,\mathbf{I_1}(e,\dot{e},n) + e^{-2}\,\mathbf{I_0}(e,\dot{e},n) - e^{-3}\,\mathbf{I_{0\text{-}}}(e,\dot{e},n)\ .$$

After some algebra, the expression (5) gives the following result for the integral $\mathbf{I_2}(e,\dot{e},n)$:

(9) $$n^{-1}\,\mathbf{I_2}(e,\dot{e},n) = (n - 2)\dot{e}(1 - e^2)[ne^2(e - \dot{e})]^{-1}\mathbf{I_{0\text{-}}}(e,\dot{e},n) + + n^{-1}\{\,2 - (n - 2)\dot{e}[e^2(e - \dot{e})]^{-1}\,\}\,\mathbf{I_0}(e,\dot{e},n) + + n^{-1}\{\,(e - \dot{e})^{-1} + (n - 2)\dot{e}[e(e - \dot{e})]^{-1}\,\}\,\mathbf{I_1}(e,\dot{e},n)\ .$$

Multiplying this equality by $ne^2(e - \dot{e})$, we shall obtain:

(10) $$e^2(e - \dot{e})\mathbf{I_2}(e,\dot{e},n) = (n - 2)(1 - e^2)\dot{e}\mathbf{I_{0\text{-}}}(e,\dot{e},n) + [2e^2(e - \dot{e}) - (n - 2)\dot{e}]\,\mathbf{I_0}(e,\dot{e},n) + [e^2 + (n - 2)e\dot{e}]\,\mathbf{I_1}(e,\dot{e},n)\ .$$

This is another linear relation, which enables us (like the equality (4)) to eliminate the integral $\mathbf{I_2}(e,\dot{e},n)$. We shall now check the validity of (10) in the cases when the power n, the eccentricity $e(u)$ and its derivative $\dot{e}(u)$ may vanish separately or simultaneously for some value $u \equiv ln\ p$ – a situation that was preliminary excluded in deriving of (10).

4.1. **Case $n \neq 0$, $e(u) = 0$, $e(u) - \dot{e}(u) = 0 \Rightarrow e(u) = \dot{e}(u) = 0$.**

The relation (10) can be written as the equality 0 = 0, i.e., it is right.

4.2. **Case $n \neq 0$, $e(u) = 0$, $e(u) - \dot{e}(u) \neq 0 \Rightarrow \dot{e}(u) \neq 0$.**

The equality (10) becomes:

(11) $$0 = (n-2)\dot{e}\mathbf{I}_{0-}(0,\dot{e},n) - (n-2)\dot{e}\mathbf{I}_0(0,\dot{e},n) .$$

If $n = 2$, (11) is trivially fulfilled. If $n \neq 2$, after the division of both sides by $(n-2)\dot{e}$, we must prove that $\mathbf{I}_{0-}(0,\dot{e},n) = \mathbf{I}_1(0,\dot{e},n)$. This was already done earlier: see equalities (81) and (82) from paper [15].

4.3. **Case $n \neq 0$, $e(u) \neq 0$, $e(u) - \dot{e}(u) = 0 \Rightarrow \dot{e}(u) = e(u) \neq 0$.**

We can write (10) in the following way:

(12) $$0 = (n-2)(1-e^2)\dot{e}\mathbf{I}_{0-}(e,\dot{e}=e,n) - (n-2)\dot{e}\mathbf{I}_0(e,\dot{e}=e,n) + (n-1)e\dot{e}\mathbf{I}_1(e,\dot{e}=e,n) .$$

Dividing by $\dot{e} \neq 0$, we arrive at the next formula, which we must to prove, in order to verify (10) in this particular case:

(13) $$(n-1)e\,\mathbf{I}_1(e,\dot{e}=e,n) = (n-2)\mathbf{I}_0(e,\dot{e}=e,n) - (n-2)(1-e^2)\mathbf{I}_{0-}(e,\dot{e}=e,n) .$$

We directly compute that:

(14) $$\mathbf{I}_1(e,\dot{e}=e,n) = \int_0^{2\pi} cos\varphi(1+e\,cos\varphi)^{n-2}\,d\varphi = \int_0^{2\pi}(1+e\,cos\varphi)^{n-2}\,d(sin\varphi) =$$

$$= sin\varphi(1+e\,cos\varphi)^{n-2}\Big|_0^{2\pi} + (n-2)e\int_0^{2\pi} sin^2\varphi(1+e\,cos\varphi)^{n-3}\,d\varphi =$$

$$= (n-2)e\int_0^{2\pi}(1-cos^2\varphi)(1+e\,cos\varphi)^{n-3}\,d\varphi = (n-2)e\int_0^{2\pi}(1+e\,cos\varphi)^{n-3}\,d\varphi \; -$$

$$- (n-2)\int_0^{2\pi} cos\varphi(1+ecos\varphi-1)(1+e\,cos\varphi)^{n-3}\,d\varphi = (n-2)e\mathbf{I}_{0-}(e,\dot{e}=e,n) -$$

$$-(n-2)\int_0^{2\pi} cos\varphi(1+e\,cos\varphi)^{n-2}\,d\varphi + (n-2)\int_0^{2\pi} cos\varphi(1+e\,cos\varphi)^{n-3}\,d\varphi =$$

$$= (n-2)e\mathbf{I}_{0-}(e,\dot{e}=e,n) - (n-2)\mathbf{I}_1(e,\dot{e}=e,n) +$$

$$+ (n-2)e^{-1}\int_0^{2\pi}(1+ecos\varphi-1)(1+e\,cos\varphi)^{n-3}\,d\varphi =$$

$$= (n-2)e\mathbf{I}_{0-}(e,\dot{e}=e,n) - (n-2)\mathbf{I}_1(e,\dot{e}=e,n) +$$

$$+ (n-2)e^{-1}\int_0^{2\pi}(1 + e\cos\varphi)^{n-2}\,d\varphi - (n-2)e^{-1}\int_0^{2\pi}(1 + e\cos\varphi)^{n-3}\,d\varphi =$$

$$= (n-2)e\mathbf{I_{0\text{-}}}(e,\dot{e} = e,n) - (n-2)\mathbf{I_1}(e,\dot{e} = e,n) + (n-2)e^{-1}\mathbf{I_0}(e,\dot{e} = e,n) -$$
$$-\ (n-2)e^{-1}\mathbf{I_{0\text{-}}}(e,\dot{e} = e,n)\ .$$

Multiplying the both sides by $e(u) \neq 0$, we obtain the equality (13), and, correspondingly, the linear relation (10) is proved.

4.4. <u>Case $n = 0$, $e(u) = 0$, $e(u) - \dot{e}(u) = 0 \Rightarrow e(u) = \dot{e}(u) = 0$.</u>

The equality (10) may be written as 0 = 0, and it is obviously fulfilled.

4.5. <u>Case $n = 0$, $e(u) = 0$, $e(u) - \dot{e}(u) \neq 0 \Rightarrow \dot{e}(u) \neq 0$.</u>

Now (10) becomes:

(15) $\quad 0 = -\,2\dot{e}\mathbf{I_{0\text{-}}}(0,\dot{e}\ ,0) + 2\dot{e}\ \mathbf{I_0}(0,\dot{e}\ ,0)$, or $\mathbf{I_{0\text{-}}}(0,\dot{e}\ ,0) = \mathbf{I_0}(0,\dot{e}\ ,0)$.

This is true, because $\mathbf{I_0}(0,\dot{e}\ ,0) = \int_0^{2\pi}(1 - \dot{e}\cos\varphi)^{-1}d\varphi = \mathbf{I_{0\text{-}}}(0,\dot{e}\ ,0)$.

4.6. <u>Case $n = 0$, $e(u) \neq 0$, $e(u) - \dot{e}(u) = 0 \Rightarrow \dot{e}(u) = e(u) \neq 0$.</u>

The linear relation(10), which must be proved, in this case becomes:

(16) $\quad 0 = -\,2(1 - e^2)\dot{e}\mathbf{I_{0\text{-}}}(e,\dot{e} = e,0) + 2\dot{e}\mathbf{I_0}(e,\dot{e} = e,0) - e^2\mathbf{I_1}(e,\dot{e} = e,0)$.

We have already computed (formula (70) from [15], multiplied by $-e^2(u) \neq 0$), that in this case we have:

(17) $\quad -\,e^2\mathbf{I_1}(e,\dot{e} = e,0) = e^3\mathbf{I_0}(e,\dot{e} = e,0)$.

Taking into account that $e(u) = \dot{e}(u)$, we also compute (substituting (17) into (16)):

(18) $\quad -\,2(1 - e^2)e\mathbf{I_{0\text{-}}}(e,\dot{e} = e,0) + 2e\mathbf{I_0}(e,\dot{e} = e,0) + e^3\mathbf{I_0}(e,\dot{e} = e,0) =$
$\quad = -\,2(1 - e^2)e\mathbf{I_{0\text{-}}}(e,\dot{e} = e,0) + e(2 + e^2)\mathbf{I_0}(e,\dot{e} = e,0)$.

For later purposes, we evaluate in an explicit form the integral $\mathbf{I_0}(e,\dot{e} = e,0)$, using the formula **858.538** from Dwight [16]:

(19) $\quad \mathbf{I_0}(e,\dot{e} = e,0) = \int_0^{2\pi}(1 + e\cos\varphi)^{-2}\,d\varphi = 2\pi[(1 - e^2)(1 - e^2)^{1/2}]^{-1}$.

In the considered case, because $\dot{e}(u) = e(u)$, the integral $\mathbf{I_0}(e,\dot{e} = e,0)$ is a function only on one independent variable, namely e. We shall differentiate this integral with respect to e. Having in mind, that differentiation and integration are linear operations, which sequence of carrying out may be interchanged, we write:

(20) $d\,\mathbf{I_0}(e,\dot{e}=e,0)/de = (d/de)[2\pi(1-e^2)^{-3/2}] = 3e(1-e^2)^{-1}2\pi(1-e^2)^{-3/2} =$
$= 3e(1-e^2)^{-1}\mathbf{I_0}(e,\dot{e}=e,0)$,

where we have applied the result (19). From the other hand:

(21) $d\,\mathbf{I_0}(e,\dot{e}=e,0)/de = (d/de)\int_0^{2\pi}(1+e\cos\varphi)^{-2}\,d\varphi = -2\int_0^{2\pi}\cos\varphi(1+e\cos\varphi)^{-3}\,d\varphi =$

$= -2e^{-1}\int_0^{2\pi}(1+e\cos\varphi-1)(1+e\cos\varphi)^{-3}\,d\varphi = -2e^{-1}\int_0^{2\pi}(1+e\cos\varphi)^{-2}\,d\varphi +$

$+ 2e^{-1}\int_0^{2\pi}(1+e\cos\varphi)^{-3}\,d\varphi = -2e^{-1}\mathbf{I_0}(e,\dot{e}=e,0) + 2e^{-1}\mathbf{I_{0\text{-}}}(e,\dot{e}=e,0)$.

Combining (20) and (21) gives:

(22) $-2e^{-1}\mathbf{I_0}(e,\dot{e}=e,0) + 2e^{-1}\mathbf{I_{0\text{-}}}(e,\dot{e}=e,0) = 3(1-e^2)^{-1}\mathbf{I_0}(e,\dot{e}=e,0)$.

Multiplication by $e(1-e^2)$ leads to:

(23) $[-2(1-e^2)-3e^2]\mathbf{I_0}(e,\dot{e}=e,0) = -2(1-e^2)\mathbf{I_{0\text{-}}}(e,\dot{e}=e,0)$.

Another multiplication of the above equality by $e(u) \neq 0$ gives that:

(24) $(2+e^2)e\mathbf{I_0}(e,\dot{e}=e,0) - 2e(1-e^2)\mathbf{I_0}(e,\dot{e}=e,0) = 0$,

that means (taking into account the results(17) and (18)) that (16) is true, and, consequently, the relation (10) is again proved.

4.7. Case $n = 0$, $e(u) \neq 0$, $e(u) - \dot{e}(u) \neq 0$.

The linear relation (10) takes the form:

(25) $e^2(e-\dot{e})\mathbf{I_2}(e,\dot{e},0) = -2(1-e^2)\dot{e}\mathbf{I_{0\text{-}}}(e,\dot{e},0) + [2e^2(e-\dot{e}) + 2\dot{e}]\mathbf{I_0}(e,\dot{e},0) +$
$+ (e^2 - 2e\dot{e})\mathbf{I_1}(e,\dot{e},0)$.

We again shall use the explicit analytical expressions for $n = 0$, derived for the integrals $\mathbf{I_{0\text{-}}}(e,\dot{e},0)$, $\mathbf{I_0}(e,\dot{e},0)$, $\mathbf{I_1}(e,\dot{e},0)$ and $\mathbf{I_2}(e,\dot{e},0)$ ([17], formulas 3a), 3b), 3c) and 3h); see also formulas (48), (49) and (50)in the paper [15]). According to the formula 3h) in [17], we are able to write for $\mathbf{I_{0\text{-}}}(e,\dot{e},0)$ the following expression:

(26) $\mathbf{I_{0\text{-}}}(e,\dot{e},0) = \boldsymbol{A}(e,\dot{e})[2(1-e^2)\dot{e}]^{-1}\{(2e^3 - 4e^5 + 2e^7 - 6e^2\dot{e} + 10e^4\dot{e} - 4e^6\dot{e} + 6e\dot{e}^2 -$
$- 5e^3\dot{e}^2 + 2e^5\dot{e}^2)[1-(e-\dot{e})^2]^{1/2} - 2(e-\dot{e})^3(1-e^2)^{5/2}\}$,

where we have used the notation (47) from paper [15] for $\boldsymbol{A}(e,\dot{e})$:

(27) $\boldsymbol{A}(e,\dot{e}) = 2\pi\dot{e}^{-2}(1-e^2)^{-3/2}[1-(e-\dot{e})^2]^{-1/2}$.

Taking into account the expression [50] from the same paper [15]:

(28) $\mathbf{I_2}(e,\dot{e},0) = \boldsymbol{A}(e,\dot{e})\{(-1+e^2+e\dot{e})[1-(e-\dot{e})^2]^{1/2} + (1-e^2)^{3/2}\}$,

we compute the left-hand-side of the relation (25), which we intent to prove:

(29) $e^2(e-\dot{e})\mathbf{I_2}(e,\dot{e},0) = e^2(e-\dot{e})\boldsymbol{A}(e,\dot{e})\{(-1+e^2+e\dot{e})[1-(e-\dot{e})^2]^{1/2}+(1-e^2)^{3/2}\} =$
$= \boldsymbol{A}(e,\dot{e})\{\,(-e^3+e^5+e^2\dot{e}-e^3\dot{e}^2)[1+(e-\dot{e})^2]^{1/2}]+e^2(e-\dot{e})(1-e^2)^{3/2}\,\}\,.$

Now we begin to evaluate the right-hand-side of (25). That is:

(30) $-2(1-e^2)\dot{e}\mathbf{I_{0-}}(e,\dot{e},0)+[2e^2(e-\dot{e})+2\dot{e}]\mathbf{I_0}(e,\dot{e},0)+(e^2-2e\dot{e})\mathbf{I_1}(e,\dot{e},0) =$
$= \boldsymbol{A}(e,\dot{e})\{\,(2e^3-4e^5+2e^7-6e^2\dot{e}+10e^4\dot{e}-4e^6\dot{e}+6e\dot{e}^2-5e^3\dot{e}^2+2e^5\dot{e}^2)\times$
$\times[1-(e-\dot{e})^2]^{1/2}-2(e-\dot{e})^3(1-e^2)^{5/2}\,\}+$
$+\boldsymbol{A}(e,\dot{e})[2e^2(e-\dot{e})+2\dot{e}]\{\,e\dot{e}[1-(e-\dot{e})^2]^{1/2}-e(e-\dot{e})(1-e^2)[1-(e-\dot{e})^2]^{1/2}+$
$+(e-\dot{e})^2(1-e^2)^{3/2}\,\}+$
$+\boldsymbol{A}(e,\dot{e})(e^2-2e\dot{e})\{\,(e-\dot{e}-e^3)[1-(e-\dot{e})^2]^{1/2}-(e-\dot{e})(1-e^2)^{3/2}\,\} =$
$= \boldsymbol{A}(e,\dot{e})\{\,(-2e^3+4e^5-2e^7+6e^2\dot{e}-10e^4\dot{e}+4e^6\dot{e}-6e\dot{e}^2+5e^3\dot{e}^2-2e^5\dot{e}^2)\times$
$\times[1-(e-\dot{e})^2]^{1/2}+(2e^3-2e^5-6e^2\dot{e}+6e^4\dot{e}+6e\dot{e}^2-6e^3\dot{e}^2-2\dot{e}^3+2e^2\dot{e}^3)(1-e^2)^{3/2}+$
$+\ (2e^4\dot{e}+2e\dot{e}^2-2e^3\dot{e}^2)[1-(e-\dot{e})^2]^{1/2}+$
$+(2e^7-2e^5-2e^2\dot{e}+6e^4\dot{e}-4e^6\dot{e}+2e\dot{e}^2-4e^3\dot{e}^2+2e^5\dot{e}^2)[1-(e-\dot{e})^2]^{1/2}+$
$+(2e^5+2e^2\dot{e}-6e^4\dot{e}-4e\dot{e}^2+6e^3\dot{e}^2+2\dot{e}^3-2e^2\dot{e}^3)(1-e^2)^{3/2}+$
$+(e^3-e^5-3e^2\dot{e}+2e\dot{e}^2+2e^4\dot{e})[1-(e-\dot{e})^2]^{1/2}+(-e^3+3e^2\dot{e}-2e\dot{e}^2)(1-e^2)^{3/2}\,\} =$
$= \boldsymbol{A}(e,\dot{e})\{\,(-e^3+e^5-e^3\dot{e}^2+e^2\dot{e})[1-(e-\dot{e})^2]^{1/2}+(e^3-e^2\dot{e})(1-e^2)^{3/2}\,\}\,.$

This coincides with the right-hand-side of (29). Consequently, the relations (25) and (10) are proved also for the case $n = 0$, $e(u) \neq 0$ and $e(u) - \dot{e}(u) \neq 0$.

To summarize the situation, we note that we have two linear relations between the integrals $\mathbf{I_2}(e,\dot{e},n)$, $\mathbf{I_1}(e,\dot{e},n)$, $\mathbf{I_0}(e,\dot{e},n)$ and $\mathbf{I_{0-}}(e,\dot{e},n)$, namely, the equalities (4) and (10). They are both valid for arbitrary integer/noninteger powers n (we consider physically reasonable values of n between -1 and $+3$), arbitrary values of the eccentricity $e(u)$ (provided that $|e(u)| < 1$) and its derivative $\dot{e}(u)$ (provided that $|e(u) - \dot{e}(u)| < 1$). To proceed further, we shall multiply (4) by $e^2(e-\dot{e})$ and also multiply (10) by $[e+(n-1)\dot{e}]$. The result is the following:

(31) $e^2(e-\dot{e})[e+(n-1)\dot{e}]\mathbf{I_2}(e,\dot{e},n) \equiv e^2(e-\dot{e})(-e+n\dot{e})\mathbf{I_0}(e,\dot{e},n) -$
$-e^2(e-\dot{e})[1+e(e-\dot{e})]\mathbf{I_1}(e,\dot{e},n) = [e+(n-1)\dot{e}](n-2)(1-e^2)\dot{e}\mathbf{I_{0-}}(e,\dot{e},n)+$
$+[e+(n-1)\dot{e}][2e^2(e-\dot{e})-(n-2)\dot{e}]\mathbf{I_0}(e,\dot{e},n)+$
$+[e+(n-1)\dot{e}][e^2+(n-2)e\dot{e}]\mathbf{I_1}(e,\dot{e},n)\,.$

The second equality in the above expression gives, in one's turn, a new linear relation between the integrals $\mathbf{I_1}(e,\dot{e},n)$, $\mathbf{I_0}(e,\dot{e},n)$ and $\mathbf{I_{0-}}(e,\dot{e},n)$:

(32) $e[-2e^2-e^4+(-2n+4)e\dot{e}+2e^3\dot{e}-(n-1)(n-2)\dot{e}^2-e^2\dot{e}^2]\mathbf{I_1}(e,\dot{e},n) =$
$= (n-2)[e\dot{e}-e^3\dot{e}+(n-1)\dot{e}^2-(n-1)e^2\dot{e}^2]\mathbf{I_{0-}}(e,\dot{e},n)+$
$+[3e^4-(n-2)e\dot{e}+(n-5)e^3\dot{e}-(n-1)(n-2)\dot{e}^2-(n-2)e^2\dot{e}^2]\mathbf{I_0}(e,\dot{e},n)\,.$

The derivation of the above result (32) supposes that the both multipliers $e(e-\dot{e}) \neq 0$ and $[e+(n-1)\dot{e}] \neq 0$. As already investigated above, the vanishing of these two multipliers does not invalidate the relations (4) and (10). Their left-hand-sides will be simply equal to zero. We stress, that our purpose in this section **4** is to eliminate the integral $\mathbf{I_0}(e,\dot{e},n)$ by means of the establishment of linear relations between $\mathbf{I_0}(e,\dot{e},n)$, $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$. Just from this point of view, we shall consider the particular cases $e(e-\dot{e})=0$ and $[e+(n-1)\dot{e}]=0$. Firstly, we shall resolve the problem namely for these two particular cases and, after that, we shall return to the equality (32).

4.8.1. Case $e(u)[e(u)-\dot{e}(u)]=0$, $[e(u)+(n-1)\dot{e}(u)]=0$, $\dot{e}(u)\neq 0$.

The relations (4) and (10) take the forms (having in mind that $-e=(n-1)\dot{e}$):

(33) $0=(2n-1)\dot{e}\mathbf{I_0}(e,\dot{e},n)-\mathbf{I_1}(e,\dot{e},n)$, or $\mathbf{I_1}(e,\dot{e},n)=(2n-1)\dot{e}\mathbf{I_0}(e,\dot{e},n)$,

(34) $0=(n-2)(1-e^2)\dot{e}\mathbf{I_{0-}}(e,\dot{e},n)-(n-2)\dot{e}\mathbf{I_0}(e,\dot{e},n)+[e^2+(n-2)e\dot{e}]\mathbf{I_1}(e,\dot{e},n)$.

Combining the above results (33) and (34), we obtain:

(35) $[(n-2)\dot{e}-(2n-1)e^2\dot{e}-(2n-1)(n-2)e\dot{e}^2]\mathbf{I_0}(e,\dot{e},n)=(n-2)(1-e^2)\dot{e}\mathbf{I_{0-}}(e,\dot{e},n)$.

As already mentioned above, $-e=(n-1)\dot{e}$ for this particular case, and we are able to write (35) as:

(36) $\dot{e}[(n-2)-(n-1)(2n-1)\dot{e}^2]\mathbf{I_0}(e,\dot{e},n)=(n-2)(1-e^2)\dot{e}\mathbf{I_{0-}}(e,\dot{e},n)$.

If $\dot{e}(u)=0$, this equality cannot be used for determination of $\mathbf{I_0}(e,\dot{e},n)$. But if $\dot{e}(u)\neq 0$, (which is the situation *in our case* !), we have:

(37) $[(n-2)-(n-1)(2n-1)\dot{e}^2]\mathbf{I_0}(e,\dot{e},n)=(n-2)(1-e^2)\mathbf{I_{0-}}(e,\dot{e},n)$.

Of course, (37) may be useful only if the multiplier $[(n-2)-(n-1)(2n-1)\dot{e}^2]\neq 0$.

It is worthy to note, that in the present case, which we consider, the left-hand-side of the relation (79) from paper [15]:

(38) $2e(e-\dot{e})\mathbf{I_1}(e,\dot{e},n)=(n-2)(e-\dot{e})(1-e^2)\mathbf{I_{0-}}(e,\dot{e},n)+$
$+(n+1)e[(e-\dot{e})^2-1]\mathbf{I_{0+}}(e,\dot{e},n)+[3e+(n-2)\dot{e}]\mathbf{I_0}(e,\dot{e},n)$

is also equal to zero. This provides another possibility to exclude the integral $\mathbf{I_0}(e,\dot{e},n)$. That is:

(39) $[3e+(n-2)\dot{e}]\mathbf{I_0}(e,\dot{e},n)=-(n-2)(e-\dot{e})(1-e^2)\mathbf{I_{0-}}(e,\dot{e},n)+$
$+(n+1)e[1-(e-\dot{e})^2]\,\mathbf{I_{0+}}(e,\dot{e},n)$.

Taking about the present particular case, the equality $e(u)=-(n-1)\dot{e}(u)$ we express (39) in the form:

(40) $(2n-1)\dot{e}\mathbf{I_0}(e,\dot{e},n)=-(n-2)n\dot{e}(1-e^2)\mathbf{I_{0-}}(e,\dot{e},n)+$
$+(n-1)(n+1)\dot{e}(1-n^2\dot{e}^2)\,\mathbf{I_{0+}}(e,\dot{e},n)$.

If $\dot{e}(u)=0$, this equality cannot be useful for the determination of $\mathbf{I_0}(e,\dot{e},n)$. But if $\dot{e}(u)\neq 0$, we have:

(41) $(2n-1)\mathbf{I_0}(e,\dot{e},n) = -n(n-2)(1-e^2)\mathbf{I_{0-}}(e,\dot{e},n) + (n-1)(n+1)(1-n^2\dot{e}^2)\,\mathbf{I_{0+}}(e,\dot{e},n)$.

If $(2n-1) = 0$ (i.e., $n = 1/2$), we are not able to eliminate the integral $\mathbf{I_0}(e,\dot{e},n)$, using the above relation (41). But nevertheless, $[(n-2)-(n-1)(2n-1)\dot{e}^2] = n-2 = -3/2 \neq 0$, and we may then use (37) to eliminate $\mathbf{I_0}(e,\dot{e},n)$. Consequently, if $\dot{e}(u) \neq 0$, the linear relations (37) and (41) ensure the elimination of $\mathbf{I_0}(e,\dot{e},n)$ in the case **4.8.1**, which particular case implies $e(u) = -(n-1)\dot{e}(u)$. *We strongly emphasize*, *that the later equality* ***must not*** *be considered*, *in general*, *as a first order ordinary differential equation for the eccentricity* $\boldsymbol{e(u)}$, *whose solution is* $\boldsymbol{e(u) = constant \times exp[-(n-1)^{-1}u]}$. Though such a situation may be (in principle) a subject of special investigation. In the present paper, we limit our computations only to concrete values $u \equiv ln\ p$ of the focal parameter p, which are able to cause troubles (e.g. singularities) in the derived by us linear relations between the seven integrals $\mathbf{I_{0-}}(e,\dot{e},n)$, $\mathbf{I_{0+}}(e,\dot{e},n)$ and $\mathbf{I_j}(e,\dot{e},n)$, ($\mathbf{j} = 0,1,\ldots,4$) (see formulas (1), (2) and (3)). We do not expect that these divergences do scope the whole range of the accretion disc. Such a pathological situation would imply that the accretion disc model itself is very wrong. So, we consider the possible “singular values” of the independent variable $u \equiv ln\ p$ as isolated points or “small” (in some sense) intervals, which do not determine the global structure of the accretion flow.

4.8.2. Case $e(u)[e(u)-\dot{e}(u)] = 0$, $[e(u)+(n-1)\dot{e}(u)] = 0$, $\dot{e}(u) = 0$.

To conclude the considerations in the paragraph **4.8**, ***we return*** to the case $\dot{e}(u) = 0$. By the hypotheses of this paragraph, $\dot{e}(u) = 0$ implies also that $e(u) = 0$. But the situation is now very trivial, simply because

$\mathbf{I_0}(0,0,n) = \int_0^{2\pi} d\varphi = 2\pi$. There does not arise the necessity to represent

the integral $\mathbf{I_0}(0,0,n)$ by means of a linear combination of the integrals $\mathbf{I_{0-}}(0,0,n)$ and $\mathbf{I_{0+}}(0,0,n)$. It is worthy to note, that in the limits $e(u) \to 0$ and $\dot{e}(u) \to 0$, the relations (37) and (41) (depending on the condition $n = 1/2$ or $n \neq 1/2$, correspondingly) give the same value 2π for $\mathbf{I_0}(0,0,n)$, although they are derived under the assumption $\dot{e}(u) \neq 0$. That is an indication for a continuous transition of the values of $\mathbf{I_0}(0,0,n)$ through the “singular” value $\dot{e}(u) = 0$. We shall not handle here in a strict mathematical manner this circumstance. Out goal is to prove that the integral $\mathbf{I_0}(0,0,n)$ is possible to be removed from the dynamical equation ([15], equation (4)) of the accretion disc.

4.8. Case $e(u)[e(u) - \dot{e}(u)] = 0$, $[e(u) + (n - 1)\dot{e}(u)] \neq 0$.

We shall prove now that under the above conditions $\dot{e}(u)$ ***cannot*** be equal to zero. If we suppose the opposite, namely that $\dot{e}(u) = 0$, then from the second condition $[e(u) + (n - 1)\dot{e}(u)] \neq 0$ it follows that $e(u) \neq 0$. But from the first equality we have $e(u) - \dot{e}(u) = 0$, or $e(u) = 0$. We obtain a ***contradiction***. Hence, $\dot{e}(u) \neq 0$. We shall consider the following subclasses:

4.9.1. Case $e(u)[e(u) - \dot{e}(u)] = 0$, $[e(u) + (n - 1)\dot{e}(u)] \neq 0$, $e(u) = 0$.

From $[e(u) + (n - 1)\dot{e}(u)] \neq 0$ it follows $(n - 1)\dot{e}(u) \neq 0$. We already just proved that $\dot{e}(u) \neq 0$. Consequently, $n \neq 1$, otherwise this case **4.9.1** will be ***inconsistent***. The relation (10) reduces to (11) and if $n \neq 2$, after dividing the both sides of (11) by $(n - 2)\dot{e}$, the result is $\mathbf{I_0}(0,\dot{e},n \neq 1, 2) = \mathbf{I_{0-}}(0,\dot{e},n \neq 1, 2)$. In the case $n = 2$, we have

$\mathbf{I_0}(0,\dot{e},2) = \int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-3}\, d\varphi = \mathbf{I_{0-}}(0,\dot{e},2)$. Therefore, for all admissible n

(i.e., $n \neq 1$) we again obtain the equality

(42) $\quad \mathbf{I_0}(0,\dot{e},n \neq 1) = \mathbf{I_{0-}}(0,\dot{e},n \neq 1)$.

4.9.2. Case $e(u)[e(u) - \dot{e}(u)] = 0$, $[e(u) + (n - 1)\dot{e}(u)] \neq 0$, $e(u) \neq 0$.

If $e(u) \neq 0$, it follows that $e(u) = \dot{e}(u) \neq 0$. The second condition implies that $n\dot{e}(u) \neq 0$. Consequently, we must reject the value $n = 0$, otherwise the case **4.9.2** will be ***inconsistent***. Direct computation shows that:

(43) $$\mathbf{I_0}(e,\dot{e} = e,n \neq 0) = \int_0^{2\pi}(1 + ecos\varphi)^{n-2}\, d\varphi = \mathbf{I_{0+}}(e,\dot{e} = e,n \neq 0) ,$$

according to the definitions (2) and (3). Of course, the equality $\mathbf{I_0}(e,\dot{e} = e,n) = \mathbf{I_{0+}}(e,\dot{e} = e,n)$ is also true for $n = 0$, out of the considered present case.

The above analysis again demonstrates the possibility to express the integral $\mathbf{I_0}(e,\dot{e},n)$ in terms of the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$, without to put in use the earlier eliminated integrals $\mathbf{I_j}(e,\dot{e},n)$ ($\mathbf{j}$ = 1,2,3,4).

4.9. Case $e(u)[e(u) - \dot{e}(u)] \neq 0$, $[e(u) + (n-1)\dot{e}(u)] = 0$.

The above conditions impose two restrictions over the power n in the viscosity law $\eta = \beta\,\Sigma^n$. From the second equality $[e(u) + (n-1)\dot{e}(u)] = 0$ it follows that:

(44) $\quad e(u) - \dot{e}(u) = -n\dot{e}(u)$ **.**

If $n = 0$, this will vanish the difference $e(u) - \dot{e}(u)$, in contradiction to the hypothesis that $e(u)[e(u) - \dot{e}(u)] \neq 0$. Moreover, if $n = 1$, the second equality also will imply that $e(u) = 0$, again in contradiction to the first requirement. The equality (44) also imposes the requirement $\dot{e}(u) \neq 0$, to avoid vanishing of the difference $e(u) - \dot{e}(u)$. Therefore, *to avoid **inconsistency*** of the case **4.10**, we must preliminary exclude the possibilities that some of the equalities $n = 0$, $n = 1$ and $\dot{e}(u) = 0$ (or combinations of them) are appearing. In our consideration, two different subclasses must be investigated separately.

4.10.1. Case $e(u)[e(u) - \dot{e}(u)] \neq 0$, $[e(u) + (n-1)\dot{e}(u)] = 0$, $n \neq 1/2$.

To establish a linear dependence between integrals $\mathbf{I}_{0-}(e,\dot{e},n)$, $\mathbf{I}_{0+}(e,\dot{e},n)$ and $\mathbf{I}_0(e,\dot{e},n)$, we shall use the relations (4) and (38), which we have already proved to be valid for arbitrary integer/noninteger powers n $(-1 \leq n \leq +3)$, $e(u)$ $(|e(u)| < 1)$ and $\dot{e}(u)$ $(|e(u) - \dot{e}(u)| < 1)$. In the present case $[e(u) + (n-1)\dot{e}(u)] = 0$, i.e., the left-hand-side of (4) is equal to zero. Multiplying (4) by $2e(u)[e(u) - \dot{e}(u)] \neq 0$, we obtain:

(45)
$$
\begin{aligned}
&2e(e-\dot{e})(-e+n\dot{e})\mathbf{I}_0(e,\dot{e},n) - [1+e(e-\dot{e})]2e(e-\dot{e})\mathbf{I}_1(e,\dot{e},n) = \\
&= 2e(e-\dot{e})(-e+n\dot{e})\mathbf{I}_0(e,\dot{e},n) - [1+e(e-\dot{e})](n-2)(e-\dot{e})(1-e^2)\mathbf{I}_{0-}(e,\dot{e},n) - \\
&- [1+e(e-\dot{e})](n+1)e[(e-\dot{e})^2-1]\mathbf{I}_{0+}(e,\dot{e},n) - \\
&- [1+e(e-\dot{e})][3e+(n-2)\dot{e}]\mathbf{I}_0(e,\dot{e},n) = 0,
\end{aligned}
$$

where we have applied the equality (38). After some algebra, the second equality may be transformed to the next form, representing the linear dependence between $\mathbf{I}_0(e,\dot{e},n)$, $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$, which we are searching for:

(46)
$$
\begin{aligned}
&[3e + 5e^3 + (n-2)\dot{e} - (n+7)e^2\dot{e} + (n+2)e\dot{e}^2]\mathbf{I}_0(e,\dot{e},n) = \\
&= (n-2)(-e + e^5 + \dot{e} + e^2\dot{e} - 2e^4\dot{e} - e\dot{e}^2 + e^3\dot{e}^2)\mathbf{I}_{0-}(e,\dot{e},n) + \\
&+ (n+1)(e - e^5 + e^2\dot{e} + 3e^4\dot{e} - e\dot{e}^2 - 3e^3\dot{e}^2 + e^2\dot{e}^3)\mathbf{I}_{0+}(e,\dot{e},n)\,.
\end{aligned}
$$

It is useful to rewrite the relation (45) into another (***equivalent*** to (46))form, which will allow us to reveal more clearly the conditions, making possible to eliminate $\mathbf{I}_0(e,\dot{e},n)$ using (45). For this purpose, we employ that under the hypothesis, valid for the considered case **4.10**, $[e(u) + (n-1)\dot{e}(u)] = 0$. Consequently, we can rewrite (44) in an equivalent form:

(47) $e(u) = -(n-1)\dot{e}(u)$, or $-e(u) = (n-1)\dot{e}(u)$,
in order to eliminate $e(u)$ from (45), obtaining for analysis a more simple expression. The result is:

(48) $(2n-1)\dot{e}[1 + 3n(n-1)\dot{e}^2]\mathbf{I_0}(e,\dot{e},n) =$
$-n(n-2)\dot{e}[1 + n(n-1)\dot{e}^2][1-(n-1)^2\dot{e}^2]\mathbf{I_{0-}}(e,\dot{e},n) -$
$-(n^2-1)\dot{e}[1 + n(n-1)\dot{e}^2](n^2\dot{e}^2-1)\mathbf{I_{0+}}(e,\dot{e},n)$.

Clearly, $\dot{e}(u)$ cancels out (we have already shown at the beginning of the case **4.10**, that $\dot{e}(u)$ cannot be equal to zero). It is also evident that (48) cannot be useful if $n = 1/2$, because its left-hand-side is then equal to zero; for this reason, we supposed in the hypotheses of the subclass **4.10.1** that $n \neq 1/2$. It remains to check is it possible that the third multiplier in the left-hand-side of (48) may happen to be zero for the given value of the argument $u \equiv ln\, p$, i.e.:

(49) $1 + 3n(n-1)\dot{e}^2(u) = 0 \quad <=> \quad 3\dot{e}^2(u)n^2 - 3\dot{e}^2(u)n + 1 = 0$.

From the above equality it is obvious that $n(n-1) < 0$, i.e., the multipliers n and $(n-1)$ have opposite signs (remember that $n \neq 0$, 1/2 and + 1). Let us consider the two alternative possibilities:

(i) $n < 0$ & $(n-1) > 0$. => We obtain a ***contradiction***, because it is impossible to have simultaneously $n < 0$ and $n > +1$.

(ii) $n > 0$ & $(n-1) < 0$ => $0 < n < +1$. Taking into account that $n \neq 1/2$, we conclude that n belongs to the union of the open intervals $(0,1/2)\mathbf{U}(1/2,1)$ if the equality (49) holds. We shall use the positivity of n in the deriving of the next inequalities.

Having in mind the equalities (44) and (47) (valid for the case **4.10**), and the restrictions $|e(u)| < 1$ and $|e(u) - \dot{e}(u)| < 1$ (valid for any value of u), we are able to rewrite them in the following way:

(50) $n\,|\dot{e}(u)| < 1$,

(51) $(1-n)\,|\dot{e}(u)| < 1$.

Summation of these two inequalities immediately gives:

(52) $|\dot{e}(u)| < 2$.

Let us consider (49) as a quadratic equation for the power n. The discriminant of this equation is:

(53) $\mathbf{D} = 9\dot{e}^4(u) - 12\dot{e}^2(u) \equiv \dot{e}^4(u)[9 - 12\dot{e}^{-2}(u)]$.

We are seeking only for real solutions of the equation (49), which means that $\mathbf{D} \geq 0$. Therefore:

(54) $\dot{e}^2(u) \geq 4/3 \quad => \quad |\dot{e}(u)| \geq 2/\sqrt{3} > 1$.

Combining (52) and (54) leads to the limitations:

(55) $1 < 2/\sqrt{3} \leq |\dot{e}(u)| < 2$.

The two solutions of the equation (49) are:

(56) $n_{1,2} = (6\dot{e}^2)^{-1}(6\dot{e}^2 \pm \sqrt{\mathbf{D}})$.

Strictly speaking, the value $n = +1$ is already excluded and the discriminant **D** must be positive. Correspondingly, (55) has to be corrected as $2/\sqrt{3} < |\dot{e}(u)|$. Moreover, n belongs to the union $(0,1/2)\mathbf{U}(1/2,1)$, and if we choose in (56) the sign "+", the solution for n will be greater than +1. To avoid the contradiction, we must select the sign "–". Then the solution of the equation (49) is:

(57) $n = 1 - [1/4 - (3\dot{e}^2)^{-1}]^{1/2}$.

Obviously, $n < +1$, and the condition $n > 0$ is also satisfied, because $\dot{e}^{-2} > -9/4$. It seems that there is a *possibility* to exist a relation between $\dot{e}(u)$ and the power n belonging to $(0,1/2)\mathbf{U}(1/2,1)$, namely, the equality (57), which implies nullification of the multiplier $[1 + 3n(n-1)\dot{e}^2(u)]$ in the linear relation (48). We shall now show that such a *possibility*, in fact, ***cannot be realized***. Let us accept that the equality (49) is realized. We compute the common factor $[1 + n(n-1)\dot{e}^2(u)]$, which presents in the both terms in the right-hand-side of (48).

(58) $[1 + n(n-1)\dot{e}^2(u)] \equiv [1 + 3n(n-1)\dot{e}^2(u)] - 2n(n-1)\dot{e}^2 = -2n(n-1)\dot{e}^2 \neq 0,$

because $n \neq 0, +1$ and $\dot{e}(u) \neq 0$. Then the equality (48) takes the form (under the condition (49)):

(59) $0 = -n(n-2)[1-(n-1)^2\dot{e}^2]\mathbf{I}_{0-}(e,\dot{e},n) - (n^2-1)(n^2\dot{e}^2-1)\mathbf{I}_{0+}(e,\dot{e},n),$

where a cancellation by $\dot{e}[1 + n(n-1)\dot{e}^2] \neq 0$ is performed. Expressing $\dot{e}^2(u)$ through the power n, using again (49), we have:

(60) $\dot{e}^2(u) = -[3n(n-1)]^{-1}$, (remember that in the considered case n belongs to the union of the open intervals $(0,1/2)\mathbf{U}(1/2,1)$).

Substitution of this equality into (59) leads to:

(61) $(n-2)(4n-1)\mathbf{I}_{0-}(e,\dot{e},n) + (n+1)(-4n+3)\,\mathbf{I}_{0+}(e,\dot{e},n) = 0$.

At the present stage we shall use a result, which will be proved in a forthcoming paper; namely, the integrals $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$ are *linearly independent* functions on $e(u)$ and $\dot{e}(u)$. This is the reason, for which we prefer to eliminate the integrals $\mathbf{I}_4(e,\dot{e},n)$, $\mathbf{I}_2(e,\dot{e},n)$, $\mathbf{I}_1(e,\dot{e},n)$ and $\mathbf{I}_0(e,\dot{e},n)$ from the dynamical equation ([15], equation (4)) of the accretion disc, and to remain the integrals $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$. The situation with the integral $\mathbf{I}_3(e,\dot{e},n)$ is at present unclear. The linear independence between $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$ implies that the coefficients before these two integrals in the linear relation (61) are identically equal to zero:

(62) $(n-2)(4n-1) = 0 \quad \Rightarrow \quad n = 1/4$, because $n \neq 2$,

(63) $(n+1)(-4n+3) = 0$.

From the later equation (63) we have two different possible solutions: (i) $n = -1$, or (ii) $n = 3/4$. Both they contradict to the solution $n = 1/4$, implied from (62). But (62) and (63) must hold simultaneously. Consequently, this controversial situation means that $[1 + 3n(n - 1)\dot{e}^2]$ ***cannot be equal to zero*** for any value $u \equiv ln\ p$ and the *possible* relation between n and $\dot{e}(u)$, admitted by the equality (57) also ***cannot be realized***. As a final result, we conclude that $(2n - 1)\dot{e}[1 + 3n(n - 1)\dot{e}^2] \neq 0$. This closes the consideration of the case **4.10.1**.

4.10.2. Case $e(u)[e(u) - \dot{e}(u)] \neq 0$, $[e(u) + (n - 1)\dot{e}(u)] = 0$, $n = 1/2$.

We directly compute from the definition (3) (for $\mathbf{j} = 0$) that:

$$(64)\qquad \mathbf{I_0}[e = (1/2)\dot{e},\dot{e},1/2] = \int_0^{2\pi}[1 - (n-1)\dot{e}cos\varphi]^{-3/2}(1 - n\dot{e}cos\varphi)^{-3/2}\, d\varphi =$$

$$= \int_0^{2\pi}[1 - (\dot{e}^2/4)cos^2\varphi]^{-3/2}\, d\varphi > 0\ ,$$

where we have used that for $n = 1/2$ we have $e(u) = (1/2)\dot{e}(u)$ (formula (47)) and $(e - \dot{e}) = - n\dot{e}(u) = - (1/2)\dot{e}(u)$ (formula (44)).

It is known from the analysis, that the definition of the complete elliptic integral of the second kind $\mathbf{E}(k^2)$ is given by ([16], formula **771.**):

$$(65)\qquad \mathbf{E}(\pi/2, k) \equiv \mathbf{E}(k^2) = \int_0^{2\pi}(1 - k^2sin^2\varphi)^{1/2}\, d\varphi\ .$$

The condition (55) ensures that $(\dot{e}^2/4) < 1$ and then:

$$(66)\qquad \mathbf{I_0}[e = (1/2)\dot{e},\dot{e},1/2] = 2[1 - (\dot{e}^2/4)]^{-1}\mathbf{E}(\dot{e}^2/4) + \\ + 2[1 - (\dot{e}^2/4)]^{-1/2}\mathbf{E}\{-(\dot{e}^2/4)[1 - (\dot{e}^2/4)]^{-1}\} \underset{\dot{e}(u) \to 0}{\longrightarrow} 2\pi\ .$$

We only among the other things show the above formula, in order to manifest the existence of an explicit analytical expression for the integral $\mathbf{I_0}[e = (1/2)\dot{e},\dot{e},1/2]$. We shall not perform here the derivation of the relation (66). It turns out, that just this result goes to be of use. In fact, the hypotheses, made in the subclass **4.10.2.**, leads to the conclusion that the investigated relation (48) simply transforms to the identity $0 = 0$ if $n = 1/2$ and $e(u) = (1/2)\dot{e}(u)$. Thus, in the considered case, it becomes useless for the determination of $\mathbf{I_0}[e = (1/2)\dot{e},\dot{e},1/2]$. To see this, let us evaluate the linear relation (48) (preliminary canceling out the multiplier $\dot{e}(u) \neq 0$) in the case when $n = 1/2$:

(67) $0 = (3/4)[1-(1/4)\dot{e}^2]^2\,\mathbf{I}_{0-}(e,\dot{e},1/2) - (3/4)\,)[1-(1/4)\dot{e}^2]^2\,\mathbf{I}_{0+}(e,\dot{e},1/2)$.

The second inequality (55) $|\dot{e}(u)| < 2$ allows to cancel out the multiplier $(3/4)[1-(1/4)\dot{e}^2]^2 \neq 0$. Taking into account (47) for $n = 1/2$ (i.e., $e(u) = (1/2)\dot{e}(u)$), we rewrite (67) as:

(68) $$\mathbf{I}_{0-}[e = (1/2)\dot{e},\, \dot{e},\, 1/2] = \int_0^{2\pi}[1-(\dot{e}^2/4)cos^2\varphi]^{-3/2}[1+(1/2)\dot{e}cos\varphi]^{-1}d\varphi =$$

$$= \int_0^{2\pi}[1-(\dot{e}^2/4)cos^2\varphi]^{-3/2}[1-(1/2)\dot{e}cos\varphi]^{-1}d\varphi \equiv \mathbf{I}_{0+}[e = (1/2)\dot{e},\dot{e},1/2].$$

The above result can be also checked if we compute the difference:

(69) $$\mathbf{I}_{0-}[e = (1/2)\dot{e},\dot{e},1/2] - \mathbf{I}_{0+}[e = (1/2)\dot{e},\dot{e},1/2] =$$

$$= -\,\dot{e}\int_0^{2\pi} cos\varphi[1-(\dot{e}^2/4)cos^2\varphi]^{-5/3}d\varphi = 0.$$

The later equality is evident from the fact that $cos(\pi + \varphi) = -\,cos\varphi$.

We conclude with the investigation of that particular case, corresponding to the nullification of the factorized multiplier $e^2(e-\dot{e})[e+(n-1)\dot{e}]$. Now we return to the linear relation (32), which is useful under the condition $e^2(e-\dot{e})[e+(n-1)\dot{e}] \neq 0$. In order to obtain the factor $2e(e-\dot{e})\mathbf{I}_1(e,\dot{e},n)$, we multiply (32) by $2(e-\dot{e})$. In this way, we shall get as a factor the left-hand-side of the relation (38), i.e. we use (38) to eliminate the integral $\mathbf{I}_1(e,\, \dot{e},\, n)$. The final result is a linear relation only between the integrals $\mathbf{I}_0(e,\dot{e},n)$, $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$:

(70) $$\begin{aligned}
&[-2e^2 - e^4 + (-2n+4)e\dot{e} + 2e^3\dot{e} - (n-1)(n-2)\dot{e}^2 - e^2\dot{e}^2] \times \\
&\times (n-2)(e-\dot{e})(1-e^2)\mathbf{I}_{0-}(e,\dot{e},n) + \\
&+ [-2e^2 - e^4 + (-2n+4)e\dot{e} + 2e^3\dot{e} - (n-1)(n-2)\dot{e}^2 - e^2\dot{e}^2] \times \\
&\times e(n+1)[(e-\dot{e})^2 - 1]\mathbf{I}_{0+}(e,\dot{e},n) + \\
&+ [-2e^2 - e^4 + (-2n+4)e\dot{e} + 2e^3\dot{e} - (n-1)(n-2)\dot{e}^2 - e^2\dot{e}^2] \times \\
&[3e + (n-2)\dot{e}]\mathbf{I}_0(e,\dot{e},n) = \\
&= 2(n-2)(e-\dot{e})[e\dot{e} - e^3\dot{e} + (n-1)\dot{e}^2 - (n-1)e^2\dot{e}^2]\mathbf{I}_{0-}(e,\dot{e},n) + \\
&+ 2(e-\dot{e})[3e^4 - (n-2)e\dot{e} + (n-5)e^3\dot{e} - (n-1)(n-2)\dot{e}^2 - \\
&- (n-2)e^2\dot{e}^2]\mathbf{I}_0(e,\dot{e},n)\ .
\end{aligned}$$

After some algebra, the above equality may be put into a form, which represents $\mathbf{I}_0(e,\dot{e},n)$ as a linear combination of the integrals $\mathbf{I}_{0-}(e,\dot{e},n)$ and $\mathbf{I}_{0+}(e,\dot{e},n)$:

$$
\begin{aligned}
(71)\quad & [6e^3 + 9e^5 + 6(n-2)e^2\dot{e} + 3(n-8)e^4\dot{e} + 3(n-1)(n-2)e\dot{e}^2 - 3(2n-7)e^3\dot{e}^2 + \\
& + n(n-1)(n-2)\dot{e}^3 + 3(n-2)e^2\dot{e}^3]\mathbf{I_0}(e,\dot{e},n) = (n-2)[-2e^3 + e^5 + e^7 - 2(n-2)e^2\dot{e} + \\
& + (2n-1)e^4\dot{e} - 3e^6\dot{e} + (-n^2 + 3n - 2)e\dot{e}^2 + (n^2 - 3n - 1)e^3\dot{e}^2 + 3e^5\dot{e}^2 + n(n-1)\dot{e}^3 + \\
& + (-n^2 + n + 1)e^2\dot{e}^3 - e^4\dot{e}^3]\mathbf{I_{0-}}(e,\dot{e},n) + (n+1)[2e^3 - e^5 - e^7 + 2(n-2)e^2\dot{e} - \\
& - 2(n-3)e^4\dot{e} + 4e^6\dot{e} + (n-1)(n-2)e\dot{e}^2 + (-n^2 + 7n - 11)e^3\dot{e}^2 - 6e^5\dot{e}^2 + \\
& + 2(n-2)^2e^2\dot{e}^3 + 4e^4\dot{e}^3 - (n-1)(n-2)e\dot{e}^4 - e^3\dot{e}^4]\mathbf{I_{0+}}(e,\dot{e},n) \equiv \\
& \equiv (n-2)(1-e^2)(e-\dot{e})[-2e^2 - e^4 - 2(n-1)e\dot{e} + 2e^3\dot{e} - n(n-1)\dot{e}^2 - e^2\dot{e}^2]\mathbf{I_{0-}}(e,\dot{e},n) + \\
& + (n+1)[1-(e-\dot{e})^2]e[2e^2 + e^4 + 2(n-2)e\dot{e} - 2e^3\dot{e} + (n-1)(n-2)\dot{e}^2 + e^2\dot{e}^2] \times \\
& \times \mathbf{I_{0+}}(e,\dot{e},n).
\end{aligned}
$$

This is the linear relation for which we are seeking. Clearly, it may be relevant to the problem of the elimination of $\mathbf{I_0}(e,\dot{e},n)$, only if the multiplier before this integral is different from zero. The investigation of the case when this does not happen is much more complicated than the situation with the other integrals. We shall not investigate in the present paper the possible ineligibility to apply formula (71). We only illustrate graphically (Fig. 1) for two concrete numerical values of the power n ($n = +2.4$ and $n = -0.4$) that such a trouble really exists.

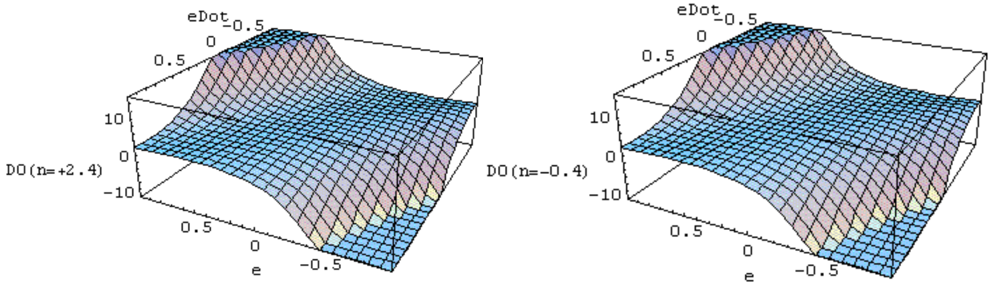

Fig. 1. Two graphics of the coefficient $\mathbf{D_0}(e,\dot{e},n) \equiv 6e^3 + 9e^5 + 6(n-2)e^2\dot{e} + 3(n-8)e^4\dot{e} + + 3(n-1)(n-2)e\dot{e}^2 - 3(2n-7)e^3\dot{e}^2 + n(n-1)(n-2)\dot{e}^3 + 3(n-2)e^2\dot{e}^3$ *for two different (arbitrary chosen) values of the power* n; ***top***: $n = +2.4$ *and* ***down***: $n = -0.4$. *Both* $e(u)$ *and* $\dot{e}(u)$ *take values from* -0.99 *to* $+0.99$.

5. Conclusions and comments

The last paragraph **4** of the present investigation, in combination with the results in the earlier paper [15], clearly demonstrate that between the seven integrals $\mathbf{I_k}(e,\dot{e},n)$, ($\mathbf{k}$ = 0-, 0+, 0, 1, 2, 3, 4) exist linear relations, which ensure the opportunity to eliminate four of them in the dynamical equation ([15], equation (4)), governing the *space* structure of the *stationary* elliptical accretion discs, according to the model of Lyubarskij et al. [7].

More concretely, we are able to express the integrals $\mathbf{I_4}(e,\dot{e},n)$, $\mathbf{I_2}(e,\dot{e},n)$, $\mathbf{I_1}(e,\dot{e},n)$ and $\mathbf{I_0}(e,\dot{e},n)$ only by means of the integrals $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$ (see definitions (1), (2) and (3)). It turn outs, that the later two integrals $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$ are linearly independent functions of the eccentricity $e(u)$ and its derivative $\dot{e}(u) \equiv de(u)/du$ for each orbit of the disc particles. This statement is quoted in advance and its proof will be given in a forthcoming paper. In such a way, the dynamical equation may be set free from some of the above cited integrals, appearing as a consequence of the azimuthal-angle averaging under the derivation of the dynamical equation in the research of Lyubarskij et al. [7]. This gives some simplification of this equation and may be eventually useful for a finding of its solution by means of analytical methods. Concerning the integral $\mathbf{I_3}(e,\dot{e},n)$, until now, we are not in a condition to eliminate it through the other integrals, *using only linear relations*. The availability of a linear dependence or independence between $\mathbf{I_{0-}}(e,\dot{e},n)$, $\mathbf{I_{0+}}(e,\dot{e},n)$ and $\mathbf{I_3}(e,\dot{e},n)$ will be a subject of a forthcoming analysis. It also remains to establish the utmost limits, under which we shall be able to attain, in our attempts to solve the dynamical equation of the elliptical disc by purely analytical methods. It is possible that this approach may turn out to be only partially successful. We hope that even in this less optimistic situation, the obtained analytical results will be useful for the more clear interpretation both of the *intermediate* calculations and the *further* necessary numerical simulations, leading to the finding of the final solution itself. Most probably, (and unfortunately), it seems that our simplifications of the dynamical equation, governing the *space* structure of the *stationary* elliptical discs ([15], equation (4) and references therein), will be relevant essentially only to the model of Lyubarskij et al. [7]. Other, much more complicated, and more realistic model of elliptical accretion discs, is developed by Ogilvie [18]. There are considered complex-valued eccentricities of the particle orbits. This mathematical approach allows to overcome the restriction of orbits sharing only a common longitude of periastron (i.e. all apse lines of the orbits are in line with each other), which limitation is an essential feature of the examined by us model of Lyubarskij et al. [7]. Unlike the later 2-dimensional analytical simulation, in the full model of Ogilvie ([18], section 3), the basic equations, governing the fluid disc, are written in 3 dimensions. In the case of elliptical discs of Lyubarskij et al. [7], the structure of the accretion flow is prescribed by ***one*** ordinary differential equation, while in the paper of Ogilvie ([18], section 4.4) ***a system of four*** ordinary differential equations must be solved. We have to take into account, that the theory, presented in [18], goes considerably beyond the previous analytical treatments of the eccentric discs. Consequently, we have to expect that the mathematical treatment of the problem will require (in principle) more complicated ways, in order to solve the structure and the dynamics of the elliptical discs. Of course, resolving the more simple case, described in [7], we hope that at least some of the established properties will be presented also (in some sense) in the real discs, observed in the nature. It will be also useful even for the process of formulation of the more realistic accretion discs models, approaching the characteristics of these objects, investigated by the methods of the observational astronomy. It is clear, that including the considerations of new details or more precisely described processes (for example, including the vertical motions in the disc), will give a better agreement between the theories and observations. But working out of models, like that of Lyubarskij et al. [7], which are not very much appropriate to approximate the really existing accretion flows, indicated by the astronomical observations, is nevertheless an unavoidable step in the direction of their more complete and perfect understanding.

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ТЪНКИ ВИСКОЗНИ ЕЛИПТИЧНИ АКРЕЦИОННИ ДИСКОВЕ С ОРБИТИ, ИМАЩИ ОБЩА ДЪЛЖИНА НА ПЕРИАСТРОНА. VI. ОПРОСТЯВАНЕ НА ДИНАМИЧНОТО УРАВНЕНИЕ

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Резюме

Ние продължаваме серията от статии, посветени на изследването и опростяването на динамичното уравнение, определящо структурата на *стационарните* елиптични акреционни дискове. Тези проучвания са в рамките, определени от модела на Любарски и др. [7]. В добавка към предишните проучвания, ние намираме още една зависимост между коефициентите на това обикновено диференциално уравнение от втори

ред, което ни дава възможност да елиминираме ефективно най-малко четири от тях. Това е в курса на нашия подход да намалим броя на тези функции, зависещи от ексцентрицитета, неговата производна и степенния показател $\boldsymbol{n}$ в закона за вискозита $\boldsymbol{\eta} = \boldsymbol{\beta}\,\boldsymbol{\Sigma}^{\boldsymbol{n}}$. Те се появяват в уравннието в течение на процеса на усредняване (т.е., при интегрирането) по азимуталния ъгъл на елиптичните орбити. На сегашния стадии на изследванията, остават все още три интеграла от указания тип. С изключение на случая на целочислени стойности на $\boldsymbol{n}$, техните аналитични решения не са известни. Във връзка с линейната зависимот или независимост на тези функции (това е предмет на нашите бъдещи проучвания), динамичното уравнение на елиптичните акреционни дискове може да бъде разцепено на една система от съответстващ брой по-прости уравнения за неизвестните ексцентрицитети на орбитите на частиците. Такъв един подход е в съответствие с нашата основна линия, прекарвана през споменатата серия от статии, да се постигне колкото се може по-голям прогрес в решаването на задачата с помощта на *чисто аналитични* методи. И само когато по-нататъшният напредък по този способ (ако крайното решение не е вече достигнато) става толкова сложен, че се оказва в безизходно положение, чак тогава да се използват числените моделирания.