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THIN VISCOUS ELLIPTICAL ACCRETION DISCS WITH ORBITS SHARING A COMMON LONGITUDE OF PERIASTRON. V. LINEAR RELATIONS BETWEEN AZIMUTHAL-ANGLE AVERAGED FACTORS IN THE DYNAMICAL EQUATION

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Abstract

We consider a model of elliptical ***stationary*** *accretion discs developed by Lyubarskij et al. [4], which have derived a second order ordinary differential equation, describing the spatial structure of these objects. This dynamical equation contains seven integrals, arising from the azimuthal averaging along the elliptical disc particle orbits. They are functions on the* ***unknown*** *eccentricity distribution* $\boldsymbol{e(u)}$*, its derivative* $\boldsymbol{\dot{e}(u) \equiv de(u)/du}$ *and the power* $\boldsymbol{n}$ *in the viscosity low* $\boldsymbol{\eta = \beta \Sigma^{n}}$*, where* $\boldsymbol{u \equiv \ln p}$*,* $\boldsymbol{p}$ *is the focal parameter of the concrete elliptical particle orbit. In the present paper, we derive linear relations between these* ***unknown*** *integrals, which may be useful to eliminate* ***three*** *of these quantities. It is also possible to eliminate even one more integral, but proving of this statement will be postponed in a forthcoming paper. The considered approach is maintained with a view to split the dynamical equation into a system of more simple differential equations.*

1. Introduction

The accretion phenomena have many impacts on the structure and evolution of large variety of astrophysical objects. Such processes may include both spherical accretion and/or accretion via discs. In the later case, the disc accretion mechanism is caused by the large angular momentum of the material, surrounding the compact body, and falling onto it as a final result. In the present investigation we shall concentrate our attention over accreting compact objects having stellar masses. As a general rule, the matter, composing the accretion flow, is supplied by another star (the so-called donor star), orbiting around the accreting compact component of the binary stellar system. In some cases, the material of the disc may be available due to a disruption by the tidal forces of a very close orbiting body. But in spite of this possible (in some sense, more “exotic”) situation, the mass of the accretion disc will disappear very soon because of the exhausting processes. These may be accumulation of mass over the surface of central star, jets and winds from the two surfaces of the disc (like the stellar winds in ordinary stars) or any other outflows removing the matter from the vicinity of the disc. Consequently, we would expect that such accretion discs may be treated as ***stationary*** objects for a time scales shorter than the corresponding time intervals for the discs existing in the close binary systems.

It is well known that the balance between the heating and cooling processes strongly determines the spatial structure and the time evolution of the accretion flows. A great variety of accretion disc models illustrates that the motions of the disc particles may essentially differ from the *Keplerian* one. This circumstance is able to change the flow so considerably, that in some parts of the disc the radial motion of matter *is not inward* (accretion), but is directed *outward* (excretion). This is the case for hot, advection-dominated accretion flows, which are usually optically thin in the radial direction. Therefore, the photons, produced at given radii, can travel long distances without being absorbed. Compton scattering of these photons heats or cools electrons at other radii of the considered accretion disc model. It may turn out to be, that at a certain radius, the Compton cooling rate is larger than the ***local*** viscous heating rate, i.e. the cooling effect is important in this situation. As pointed out by Yuan et al. [1], it is possible to obtain a self-consistent solution for the activity of an accretion disc around a black hole only when the luminosity of the disc L is less than 0,01 $L_{Eddington}$. Above this critical accretion rate, the *equilibrium* temperature of the electrons at the outer *radius* of the disc r_{out} is higher than the *virial* temperature, due to the strong Compton heating. As a result, the accretion is suppressed. Consequently, in this model, the activity of the black hole (more precisely, of its accretion flow) is expected to oscillate between an active and an inactive phase. The oscillations have time scales of the radiative time scale gas order at the outer radius r_{out} .

Another problem, associated with the description of the accretion disc structure around the black holes, is the following. The inner edges of these discs may have variable properties, if the matter inside the marginally stable orbit is magnetically connected to the disc. Then a non-zero torque is exerted on the inner disc edge and the accretion efficiency can be much higher than in the standard accretion disc model of Shakura & Sunyaev [2]. In the later case, this quantity is supposed to be equal to zero. The non-zero torque implies that, in the case of variable torque, transitions of the flow between different accretion types may be triggered [3].

In the present paper, we consider the problems related to the attempts to solve analytically the dynamical equation, governing the structure of elliptical accretion discs, rotating around a stellar mass objects. More specifically, we are dealing with the model of Lyubarskij et al. [4], which is a generalization of the work of Shakura & Sunyaev [2] to the case of elliptical accretion discs with orbits sharing a common longitude of the periastron. A very important property of the models [2] and [4] is that the trajectories of the disc particles are *Keplerian* ones. Consequently, our further conclusions cannot be applied to the above mentioned situations [1] and [3], i.e. our considerations shall avoid the cases of discs around black holes, and, especially, the disc regions too close to the central star. Such a limitation enables us also to escape the complications, related to the necessity to use general relativity for the description of disc dynamics. But these are not the only troubles, concerning the realistic treatment of the accretion flows by means of the Lyubarskij et al. model [4]. For example, angular momentum transport within young massive protoplanetary discs may be dominated by the self-gravity at the radii, where the disc is too weakly ionized to allow the development of the magnetorotational instability [5]. One important way to overcome the different problems, occurred in the theory of accretion discs, is to develop computer codes in order to perform numerical simulations of the processes in the accretion flows. Of course, such an approach may be applied for time-sequences of solutions, giving the evolution of the investigated objects. The difficulties, which arise in these searches, are very often caused by the vast volume of the needed computer capabilities. Numerical simulations of radiative processes in magnetized hot accretion discs (like these around black holes) are complicated, because the energy distributions of the particles and the photons span many orders of magnitude. The distributions may strongly depend on each other. Also, the radiative interactions behave significantly differently, depending on the energy regime. Many complications in the computational procedures are due to the enormous difference in the time-scales of the processes [6].

There are many observational evidences that the accretion discs have complex spatial structure. Photometric and spectral studies in the near infrared region of the electromagnetic spectrum have led to the identification of a new class of accretion discs, whose members have an inner optically *thick* part, separated from an outer *also* optically *thick* part by an optically *thin* gap. This is in contrast to the discs that have inner disc holes. The authors of the paper [7] Espaillat et al. take for granted that the excess of the near infrared emission above the photosphere of the star LkCa 15 is a blackbody continuum, that can only be due to the optically *thick* material in an inner disc around the star. If this result is combined with the estimation of the radius of the *inner edge* of the outer disc, it reveals a gapped structure of the accretion disc. Espaillat et al. assume that the most likely mechanism for clearing the detected gap in the evolving disc of the star LkCa 15 is the forming of planets.

Returning to the theme of numerical simulations of the accretion flows, it is worthy to note that the *two*-dimensional hydrodynamical discs are nonlinearly unstable to the formation of vortices. Once formed, these vortices survive forever. But in *three* dimensions, numerical experiments show that *only* vortices in short boxes form and survive just as in *two* dimensions. The vortices in tall boxes are unstable and are destroyed. As pointed out by Lithwick [8], the unstable vortices decay into transient turbulent-like states, that transport angular momentum *outward* at a nearly constant rate for hundreds of orbital times. In the paper [8] was derived the criterion for the vortices to survive in *three* dimensions as they do in *two* dimensions. Namely, the azimuthal extend of the considered vortex must be larger than the local scale height of the accretion disc. When this condition is violated, the vortex is unstable and decays. Lithwick [8] concludes that a vortex with a given radial extend will survive in a *three*-dimensional disc if it is sufficiently weak (vortices are longer in azimuthal than in radial extend). The weak vortices behave *two*-dimensionally even if their width is much less than their height, because they are stabilized by rotation and behave as Taylor-Proudman columns [8]. It is also important to underline that the decaying of strong vortices might be responsible for the *outward* transport of angular momentum – a condition that is required for accretion discs to accrete. Obviously, the two-dimensional analytical model of

Lyubarskij et al. [4], in which the dynamical equation is a subject of our further considerations, does not include the vortices phenomena at all. That is why, this limitation must be kept in mind when the compatibility of this model to the real accretion discs is discussed. Nevertheless, we hope that at least some of its characteristics are realistic description of the elliptical discs in the nature and there is a reason to seek for analytical solutions of this model [4]. It must be stressed that the accretion disc theory itself contains certain unresolved problems and ambiguities. In particular, turbulent viscosity is frequently used in this theory to replace the microphysical viscosity in order to accommodate the observational need in discs, that leads to enhanced transport of energy and angular momentum. In paper [9] it is shown that the mean-field approach leads not to one, but to two transport coefficients that govern the mass and angular momentum transport. The authors of the above investigation conclude that the conventional approach suffers from an inconsistent neglect of the turbulent diffusion in the surface density equation. They constrain these two new transport coefficients for the specific cases of inward, outward and zero net mass transport. Hubbard and Blackman also find that one of the new transport terms can lead to oscillations in the mean surface density, which then requires a constant or small inverse Rossby numbers for accretion discs, to maintain a monotonic power-law density [9].

The above sketched difficulties and also many other complex problems of the accretion flows theory (cited in the references of the listed below papers), unambiguously imply that we must consider models with reasonable simplifications. What assumptions we shall made depends, of course, on the accretion disc features, which we want to describe. In a series of papers [10], [11] and [12], we have investigated ***stationary*** accretion discs with ***elliptical*** shape under the assumed viscosity law $\eta = \beta\, \Sigma^{n}$, where η is the viscosity coefficient, Σ is the surface density of the disc and β is a constant. The ***ellipticity*** is the dominant property, which is assumed to characterize all the considered cases. The power n is chosen to be a free parameter, which physically reasonable values lie in the range from about –1 to about +3. The cases when n is an *integer* are already treated in the papers [11] and [13], where the dynamical equation is expressed in an analytical form. In what follows, we shall attempt to simplify this equation for *noninteger* powers n. This division of the values of the parameter n into *integer*/*noninteger* meanings has purely mathematical origin, due to our ability to solve analytically some integrals, entering into the dynamical equation. It has not physical foundations.

2. Dynamical equation for the elliptical accretion disc model

For brevity, we shall not write here in an explicit form the dynamical equation, governing the structure of the *stationary elliptical* discs *with orbits sharing a common longitude of periastron*. We only note that this matter is already studied and discussed in earlier papers [4], [10], [11], [12] and [13], and we refer the reader to these investigations. We shall remind only some definitions and assumptions, made in these publications, in order to be enough clear in the further exposition. We use the notations p and $u \equiv \ln p$ for the focal parameter of each particle trajectory and its logarithm, respectively. We shall consider the power n in the viscosity law $\eta = \beta\,\Sigma^n$ as a constant parameter for each concrete considered model. This assumption means that n is the same constant throughout the disc, i.e. its derivative with respect to p (or $u \equiv \ln p$) is equal to zero. We also assume that n may be either integer or noninteger, ranging between ≈ -1 and $\approx +3$, depending on the considered accretion disc model (but remaining as a constant in the framework of the model!). By $e \equiv e(u)$ we denote the eccentricity of the elliptical orbit of the particle, and by $\dot{e} \equiv \dot{e}(u) \equiv de/du \equiv de/d\ln p$ we understand the corresponding ordinary derivative. As it is already proved in [13], the dynamical equation, governing the structure of the accretion flow, is a second order ***homogeneous*** ordinary differential equation. Consequently, our problem is to simplify the coefficients, entering as multipliers into the two terms containing $\ddot{e}(u)$ and $\dot{e}(u)$ separately. In the paper [13] it is suggested that the procedure of the simplification may probably involve finding of linear relations between the following seven integrals $\mathbf{I_{0-}}$, $\mathbf{I_{0+}}$, $\mathbf{I_0}$, $\mathbf{I_1}$, $\mathbf{I_2}$, $\mathbf{I_3}$ and $\mathbf{I_4}$:

(1) $$\mathbf{I}_{0-}(e,\dot{e},n) \equiv \int_0^{2\pi} (1 + e\cos\varphi)^{n-3}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi\ ,$$

(2) $$\mathbf{I}_{0+}(e,\dot{e},n) \equiv \int_0^{2\pi} (1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+2)}\, d\varphi\ ,$$

(3) $$\mathbf{I_j}(e,\dot{e},n) \equiv \int_0^{2\pi} (\cos\varphi)^{\mathbf{j}}(1 + e\cos\varphi)^{n-2}[1 + (e - \dot{e})\cos\varphi]^{-(n+1)}\, d\varphi\ ;\ \mathbf{j} = 0, 1, 2, 3, 4,$$

where φ is the azimuthal angle over which the averaging is performed ([4], [10]). Using the above notations, we can write the dynamical equation of the elliptical disc in the following form ([10], [13]):

$$(4) \qquad \sum_{i,k} \mathbf{A}_{ik}(e,\dot{e},n)\, \mathbf{I}_i(e,\dot{e},n)\, \mathbf{I}_k(e,\dot{e},n)\, \ddot{e} + \sum_{l,m} \mathbf{B}_{lm}(e,\dot{e},n)\, \mathbf{I}_l(e,\dot{e},n)\, \mathbf{I}_m(e,\dot{e},n)\, \dot{e} = 0\ ,$$

where the indices i, k, l and m independently take meanings 0-, 0+, 0, 1, 2, 3 and 4. Our base line in the present paper is to obtain ***linear*** relations between the integrals $\mathbf{I_i}(e,\dot{e},n)$, (i = 0-, 0_+, 0, 1, 2, 3, 4), which will allow us to reduce the number of these integrals in the *homogeneous* ordinary differential equation (4). As already mentioned in [13], this is another approach to perform simplifications of the considered dynamical equation (4). In the forthcoming calculations we suppose at first that, ***by hypothesis***, n, $n-1$ and $n-2$ are not equal to zero. Consequently, if these quantities appear as factors in the denominators of the derived intermediate and final expressions, they (by themself) cannot cause divergences of the results. After that, we include considerations of these particular cases, in order to ensure the completeness of the task solution. In the next paragraph we shall deduce expressions which will enable us to eliminate three of the above seven integrals, namely: $\mathbf{I_4}$, $\mathbf{I_2}$ and $\mathbf{I_1}$. In following papers, we shall also remove the integral $\mathbf{I_0}$ and shall discuss the linear independence of the remaining three integrals $\mathbf{I_3}$, $\mathbf{I_{0-}}$ and $\mathbf{I_{0+}}$. We stress that all the integrals are considered to be functions on $e(u)$, $\dot{e}(u)$ and n. The later quantity n has the same value for the entire area of the elliptical accretion disc, i.e. n does not depend on the focal parameter p ($u \equiv ln\ p$). Of course, for other *concrete* models n may have different (but also constant) values. As a final result, the integrals $\mathbf{I}_i(e,\dot{e},n)$ (i = 0-, 0+, 0, 1, 2, 3, 4) will depend on u, but in the following calculations we shall consider e and $\dot{e}$ as ***independent*** variables, having however in mind, that $\dot{e}(u) \equiv de(u)/du$. The later circumstance must be taken into account when a differentiation of the integrals (1) – (3) should be performed.

3. Linear relations between integrals

During the next calculations we shall use the well-known trivial relation $cos^2\varphi + sin^2\varphi = 1$ (or, equivalently: $cos^2\varphi = 1 - sin^2\varphi$), valid for all $0 \leq \varphi \leq 2\pi$. We also remember, that according to the original work of Lyubarskij et al. [4], we must limit our investigation to the cases when not only $|e(u)| < 1$, but also the conditions $|\dot{e}(u)| < 1$ and $|e(u) - \dot{e}(u)| < 1$ are fulfilled (see the denominators of the expressions in *Appendix A* of [4]). Such restrictions probably preserve us from the much more complicated situation, when shock waves induced/generated by the singularities must be taken into account in the considered accretion disc model.

3.1. Elimination of the integral $\mathbf{I}_4(e,\dot{e},n)$

According to the definition (3), we have that:

$$(5)\qquad \mathbf{I}_4(e,\dot{e},n) \equiv \int_0^{2\pi}(\cos\varphi)^4\,(1+e\cos\varphi)^{n-2}[1+(e-\dot{e})\cos\varphi]^{-(n+1)}\,d\varphi =$$

$$= e^{-1}\int_0^{2\pi}\cos^3\varphi[(1+e\cos\varphi)-1](1+e\cos\varphi)^{n-2}[1+(e-\dot{e})\cos\varphi]^{-(n+1)}\,d\varphi =$$

$$= e^{-1}\int_0^{2\pi}\cos^3\varphi(1+e\cos\varphi)^{n-1}[1+(e-\dot{e})\cos\varphi]^{-(n+1)}\,d\varphi -$$

$$- e^{-1}\int_0^{2\pi}\cos^3\varphi(1+e\cos\varphi)^{n-2}[1+(e-\dot{e})\cos\varphi]^{-(n+1)}\,d\varphi\,.$$

The second integral is equal to $\mathbf{I}_3(e,\dot{e},n)$ (see definition (3)). Applying the relation $\cos^2\varphi = 1 - \sin^2\varphi$, we obtain:

$$(6)\qquad \mathbf{I}_4(e,\dot{e},n) = -\,e^{-1}\,\mathbf{I}_3(e,\dot{e},n) + e^{-1}\int_0^{2\pi}\cos\varphi(1-\sin^2\varphi)(1+e\cos\varphi)^{n-1}\times$$

$$\times[1+(e-\dot{e})\cos\varphi]^{-(n+1)}\,d\varphi = -\,e^{-1}\,\mathbf{I}_3(e,\dot{e},n) + e^{-1}\int_0^{2\pi}\cos\varphi(1+e\cos\varphi)^{n-1}\times$$

$$\times[1+(e-\dot{e})\cos\varphi]^{-(n+1)}\,d\varphi + [e(e-\dot{e})]^{-1}\int_0^{2\pi}\cos\varphi\,\sin\varphi(1+e\cos\varphi)^{n-1}\times$$

$$\times[1+(e-\dot{e})\cos\varphi]^{-(n+1)}\,d[1+(e-\dot{e})\cos\varphi]\,.$$

The second integral in (6) can be immediately expressed through the integrals $\mathbf{I}_1(e,\dot{e},n)$ and $\mathbf{I}_2(e,\dot{e},n)$, again using the definitions (3):

$$(7)\qquad \int_0^{2\pi}\cos\varphi(1+e\cos\varphi)^{n-1}[1+(e-\dot{e})\cos\varphi]^{-(n+1)}\,d\varphi = \mathbf{I}_1(e,\dot{e},n) + e\mathbf{I}_2(e,\dot{e},n)\,.$$

In deriving of the third summand in the relation (6), we have used that the difference $(e-\dot{e})$ does not depend on the azimuthal angle φ. Consequently:

$$(8)\qquad \mathbf{I}_4(e,\dot{e},n) = e^{-1}\mathbf{I}_1(e,\dot{e},n) + \mathbf{I}_2(e,\dot{e},n) - e^{-1}\mathbf{I}_3(e,\dot{e},n) - [ne(e-\dot{e})]^{-1}\int_0^{2\pi}\cos\varphi\,\sin\varphi\,\times$$

$$\times(1+e\cos\varphi)^{n-1}\,d\{[1+(e-\dot{e})\cos\varphi]^{-n}\}\,.$$

Let us consider now the forth term in the above equality. Integrating by parts, we obtain:

(9) $$-[ne(e-\dot{e})]^{-1}\int_0^{2\pi} cos\varphi\ sin\varphi(1+ecos\varphi)^{n-1}\,d\{\,[1+(e-\dot{e})cos\varphi]^{-n}\,\} = -[ne(e-\dot{e})]^{-1}\times$$

$$\times\{\,cos\varphi\ sin\varphi(1+ecos\varphi)^{n-1}[1+(e-\dot{e})cos\varphi]^{-n}\,\Big|_0^{2\pi} -$$

$$-\int_0^{2\pi}[1+(e-\dot{e})cos\varphi]^{-n}\,d\{\,[\,cos\varphi\ sin\varphi(1+ecos\varphi)^{n-1}\}\,\} =$$

$$= [ne(e-\dot{e})]^{-1}\{-\int_0^{2\pi} sin^2\varphi(1+ecos\varphi)^{n-1}[1+(e-\dot{e})cos\varphi]^{-n}\,d\varphi\ +$$

$$+\int_0^{2\pi} cos^2\varphi(1+ecos\varphi)^{n-1}[1+(e-\dot{e})cos\varphi]^{-n}\,d\varphi\ -$$

$$-(n-1)e\int_0^{2\pi} cos\varphi\ sin^2\varphi(1+ecos\varphi)^{n-2}[1+(e-\dot{e})cos\varphi]^{-n}\,d\varphi\,\} =$$

$$= [ne(e-\dot{e})]^{-1}\{\,2\int_0^{2\pi} cos^2\varphi(1+ecos\varphi)^{n-1}[1+(e-\dot{e})cos\varphi]^{-n}\,d\varphi\ -$$

$$-\int_0^{2\pi}(1+ecos\varphi)^{n-1}[1+(e-\dot{e})cos\varphi]^{-n}\,d\varphi -$$

$$-(n-1)e\int_0^{2\pi} cos\varphi(1+ecos\varphi)^{n-2}[1+(e-\dot{e})cos\varphi]^{-n}\,d\varphi\ +$$

$$+(n-1)e\int_0^{2\pi} cos^3\varphi(1+ecos\varphi)^{n-2}[1+(e-\dot{e})cos\varphi]^{-n}\,d\varphi\,\}\ .$$

In the above derivation we have used the relation $cos^2\varphi + sin^2\varphi = 1$ and trivially following from it simple equality:

(10) $-sin^2\varphi + cos^2\varphi = 2\,cos^2\varphi - 1$.

Like the expression (7), we shall preliminary compute several auxiliary relations, which will help us further simplify the expression (9):

(11) $$\int_0^{2\pi}(1+ecos\varphi)^{n-2}[1+(e-\dot{e})cos\varphi]^{-n}\,d\varphi\ =$$

$$= \int_0^{2\pi}[1+(e-\dot{e})cos\varphi](1+ecos\varphi)^{n-2}[1+(e-\dot{e})cos\varphi]^{-(n+1)}\,d\varphi\ =$$

$$= \mathbf{I_0}(e,\dot{e},n) + (e-\dot{e})\mathbf{I_1}(e,\dot{e},n)\ .$$

By analogy with the above computation, we multiply ***both*** the nominator and the denominator of the integrals by $[1 + (e - \dot{e})cos\varphi]$. The assumed condition $|e(u) - \dot{e}(u)| < 1$ for all u imply that this expression is never equal to zero. By a fully similar way, we evaluate the following integrals:

(12) $$\int_0^{2\pi} cos\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n}\, d\varphi = \mathbf{I_1}(e,\dot{e},n) + (e - \dot{e})\mathbf{I_2}(e,\dot{e},n)\ ,$$

(13) $$\int_0^{2\pi} cos^2\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n}\, d\varphi = \mathbf{I_2}(e,\dot{e},n) + (e - \dot{e})\mathbf{I_3}(e,\dot{e},n)\ ,$$

(14) $$\int_0^{2\pi} cos^3\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n}\, d\varphi = \mathbf{I_3}(e,\dot{e},n) + (e - \dot{e})\mathbf{I_4}(e,\dot{e},n)\ ,$$

where, of course, we have used definitions (3). Then, we continue the transformation of the right-hand side of (9):

(15) $$-[ne(e - \dot{e})]^{-1}\int_0^{2\pi} cos\varphi\ sin\varphi(1 + ecos\varphi)^{n-1}\, d\{[1 + (e - \dot{e})cos\varphi]^{-n}\} =$$

$$= [ne(e - \dot{e})]^{-1}\{\ 2\int_0^{2\pi} cos^2\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n}\, d\varphi\ +$$

$$+ 2e\int_0^{2\pi} cos^3\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n}\, d\varphi\ -$$

$$-\int_0^{2\pi}(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n}\, d\varphi\ -$$

$$-\ e\int_0^{2\pi} cos\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n}\, d\varphi\ +$$

$$+ (-\ ne + e)\int_0^{2\pi} cos\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n}\, d\varphi\ +$$

$$+ (ne - e)\int_0^{2\pi} cos^3\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n}\, d\varphi\ \}\ =$$

$$= [ne(e - \dot{e})]^{-1}\{\ 2\mathbf{I_2}(e,\dot{e},n) + 2(e - \dot{e})\mathbf{I_3}(e,\dot{e},n) + 2e\mathbf{I_3}(e,\dot{e},n) + 2e(e - \dot{e})\mathbf{I_4}(e,\dot{e},n) -$$
$$-\ \mathbf{I_0}(e,\dot{e},n) - (e - \dot{e})\mathbf{I_1}(e,\dot{e},n) - e\mathbf{I_1}(e,\dot{e},n) - e(e - \dot{e})\mathbf{I_2}(e,\dot{e},n) + (-\ ne + e)\mathbf{I_1}(e,\dot{e},n) +$$
$$+ (-\ ne + e)(e - \dot{e})\mathbf{I_2}(e,\dot{e},n) + (ne - e)\mathbf{I_3}(e,\dot{e},n) + (ne - e)(e - \dot{e})\ \mathbf{I_4}(e,\dot{e},n)\ \}\ .$$

Substituting this result into (8) and multiplying by $ne(e - \dot{e})$, we obtain the following expression for the integral $\mathbf{I_4}(e,\dot{e},n)$:

(16) $e(e-\dot{e})\mathbf{I_4}(e,\dot{e},n) = \mathbf{I_0}(e,\dot{e},n) + [e + (n-1)\dot{e}]\mathbf{I_1}(e,\dot{e},n) - 2\mathbf{I_2}(e,\dot{e},n) -$
$- [3e + (n-2)\dot{e}]\mathbf{I_3}(e,\dot{e},n).$

In the above derivation we have supposed that $n \neq 0$, $e(u) \neq 0$ and $[e(u) - \dot{e}(u)] \neq 0$. But the linear relation makes sense even if some of these conditions are not fulfilled. We shall now check the validity of (16) for these particular cases. In the next, we suppose that u is a certain value of the logarithm of the focal parameter p, for which we have $e(u) = 0$, or $[e(u) - \dot{e}(u)] = 0$, or both equalities $e(u) = 0$ and $[e(u) - \dot{e}(u)] = 0$ hold. In other words, the cases *integer/noninteger n*, *zero/nonzero e*(u) and *zero/nonzero* $[e(u) - \dot{e}(u)]$ give $2^3 = 8$ combinations. The relation (16) is until now proved only for one case, namely $n \neq 0$, $e(u) \neq 0$ and $[e(u) - \dot{e}(u)] \neq 0$ ***simultaneously***. We shall now prove (16) for the rest ***seven*** cases, which may be considered (in some sense) as certain particular exceptional situations.

3.1.1. Case $n \neq 0$, $e(u) = 0$, $e(u) - \dot{e}(u) = 0$ => $e(u) = \dot{e}(u) = 0$.

The linear relation (16) can be written as:

(17) $0 = \mathbf{I_0}(0,0,n) - 2\,\mathbf{I_2}(0,0,n).$

We immediately compute that $\mathbf{I_0}(0,0,n) = \int_0^{2\pi} d\varphi = 2\pi$ and $\mathbf{I_2}(0,0,n) = \int_0^{2\pi} cos^2\varphi\, d\varphi = \pi$.

Obviously, (17) is fulfilled.

3.1.2.1. Case $n \neq 0$, $n \neq 1$, $e(u) = 0$, $e(u) - \dot{e}(u) \neq 0$ => $\dot{e}(u) \neq 0$.

The relation (16) now becomes:

(18) $0 = \mathbf{I_0}(0,\dot{e},n) + (n-1)\dot{e}\mathbf{I_1}(0,\dot{e},n) - 2\mathbf{I_2}(0,\dot{e},n) - (n-2)\dot{e}\mathbf{I_3}(0,\dot{e},n)$.

We compute directly that:

(19) $\mathbf{I_3}(0,\dot{e},n) = \int_0^{2\pi} cos^3\varphi(1-\dot{e}cos\varphi)^{-(n+1)}\,d\varphi = -\,\dot{e}^{-1}\int_0^{2\pi} cos^2\varphi[(1-\dot{e}cos\varphi)-1] \times$

$\times (1-\dot{e}cos\varphi)^{-(n+1)}\,d\varphi = \dot{e}^{-1}\int_0^{2\pi} cos^2\varphi(1-\dot{e}cos\varphi)^{-(n+1)}\,d\varphi\; -$

$-\,\dot{e}^{-1}\int_0^{2\pi} cos^2\varphi(1-\dot{e}cos\varphi)^{-n}\,d\varphi = \dot{e}^{-1}\mathbf{I_2}(0,\dot{e},n) - \dot{e}^{-1}\int_0^{2\pi}(1-sin^2\varphi)(1-\dot{e}cos\varphi)^{-n}\,d\varphi\; =$

$= \dot{e}^{-1}\mathbf{I_2}(0,\dot{e},n) - \dot{e}^{-1}\int_0^{2\pi}(1-\dot{e}cos\varphi)^{-n}\,d\varphi + \dot{e}^{-2}\int_0^{2\pi} sin\varphi(1-\dot{e}cos\varphi)^{-n}\,d(1-\dot{e}cos\varphi)\; =$

$$= \dot{e}^{-1}\mathbf{I_2}(0,\dot{e},n) - \dot{e}^{-1}\mathbf{I_0}(0,\dot{e},n) + \mathbf{I_1}(0,\dot{e},n) + [(-n+1)\dot{e}^2]^{-1}\int_0^{2\pi} sin\varphi d[(1-\dot{e}cos\varphi)^{-(n-1)}] =$$

$$= - \dot{e}^{-1}\mathbf{I_0}(0,\dot{e},n) + \mathbf{I_1}(0,\dot{e},n) + \dot{e}^{-1}\mathbf{I_2}(0,\dot{e},n) - [(n-1)\dot{e}^2]^{-1} \times$$

$$\times \{ sin\varphi[(1-\dot{e}cos\varphi)^{-(n-1)}] \Big|_0^{2\pi} - \int_0^{2\pi} cos\varphi[(1-\dot{e}cos\varphi)^{-(n-1)}]\, d\varphi \} =$$

$$= - \dot{e}^{-1}\mathbf{I_0}(0,\dot{e},n) + \mathbf{I_1}(0,\dot{e},n) + \dot{e}^{-1}\mathbf{I_2}(0,\dot{e},n) + [(n-1)\dot{e}^2]^{-1}\int_0^{2\pi} cos\varphi(1-2\dot{e}cos\varphi +$$

$$+ \dot{e}^2cos^2\varphi)(1-\dot{e}cos\varphi)^{-(n+1)}\, d\varphi = - \dot{e}^{-1}\mathbf{I_0}(0,\dot{e},n) + \mathbf{I_1}(0,\dot{e},n) + \dot{e}^{-1}\mathbf{I_2}(0,\dot{e},n) +$$

$$+[(n-1)\dot{e}^2]^{-1}\mathbf{I_1}(0,\dot{e},n) - 2[(n-1)\dot{e}]^{-1}\mathbf{I_2}(0,\dot{e},n) + (n-1)^{-1}\mathbf{I_3}(0,\dot{e},n)\ .$$

Consequently, for $n \neq 1$ (by supposition), after multiplying the both sides of (19) by $\dot{e}(n-1)$, we have:

(20) $$- (n-1)\mathbf{I_0}(0,\dot{e},n) + [\dot{e}^{-1} + (n-1)\dot{e}]\mathbf{I_1}(0,\dot{e},n) + (n-3)\mathbf{I_2}(0,\dot{e},n) - - (n-2)\dot{e}\mathbf{I_3}(0,\dot{e},n) = 0\ .$$

By direct computation we also get:

(21) $$\mathbf{I_2}(0,\dot{e},n) = \int_0^{2\pi} cos^2\varphi(1-\dot{e}cos\varphi)^{-(n+1)}\, d\varphi = - \dot{e}^{-1}\int_0^{2\pi} cos\varphi[(1-\dot{e}cos\varphi) - 1] \times$$

$$\times (1-\dot{e}cos\varphi)^{-(n+1)}\, d\varphi = \dot{e}^{-1}\int_0^{2\pi} cos\varphi(1-\dot{e}cos\varphi)^{-(n+1)}\, d\varphi\ -$$

$$- \dot{e}^{-1}\int_0^{2\pi} cos\varphi(1-\dot{e}cos\varphi)^{-n}\, d\varphi = \dot{e}^{-1}\mathbf{I_1}(0,\dot{e},n) - \dot{e}^{-1}\int_0^{2\pi}(1-\dot{e}cos\varphi)^{-n}\, d\, sin\varphi\ =$$

$$= \dot{e}^{-1}\mathbf{I_1}(0,\dot{e},n) - \dot{e}^{-1}\{ sin\varphi[(1-\dot{e}cos\varphi)^{-n}] \Big|_0^{2\pi} - \int_0^{2\pi} sin\varphi\, d\{ [(1-\dot{e}cos\varphi)^{-n}] \} \} =$$

$$= \dot{e}^{-1}\mathbf{I_1}(0,\dot{e},n) - n\int_0^{2\pi}(1-cos^2\varphi)[(1-\dot{e}cos\varphi)^{-(n+1)}]\, d\varphi\ =$$

$$= \dot{e}^{-1}\mathbf{I_1}(0,e,n) - n[\mathbf{I_0}(0,e,n) - \mathbf{I_2}(0,e,n)]\ .$$

Therefore:

(22) $(n-1)\mathbf{I_2}(0,e,n) = n\mathbf{I_0}(0,e,n) - \dot{e}^{-1}\mathbf{I_1}(0,e,n)$, or

(23) $(n-3)\mathbf{I_2}(0,e,n) = n\mathbf{I_0}(0,e,n) - \dot{e}^{-1}\mathbf{I_1}(0,e,n) - 2\mathbf{I_2}(0,e,n)$.

Substituting this result for $(n-3)\mathbf{I_2}(0,e,n)$ into the relation (20), we shall obtain the sought equality (18).

3.1.2.2. Case $n = 1$, $e(u) = 0$, $e(u) - \dot{e}(u) \neq 0 \Rightarrow \dot{e}(u) \neq 0$.

The relation (16) becomes:

(24) $$0 = \mathbf{I_0}(0,\dot{e},1) - 2\mathbf{I_2}(0,\dot{e},1) + \dot{e}\mathbf{I_3}(0,\dot{e},1) .$$

We compute directly that:

(25) $$\mathbf{I_3}(0,\dot{e},1) = \int_0^{2\pi} cos^3\varphi(1 - \dot{e}cos\varphi)^{-2}\, d\varphi = \dot{e}^{-3}\int_0^{2\pi}(- \dot{e}^3cos^3\varphi)(1 - \dot{e}cos\varphi)^{-2}\, d\varphi =$$

$$= - \dot{e}^{-3}\int_0^{2\pi}(1 - 3\dot{e}cos\varphi + 3\dot{e}^2cos^2\varphi - \dot{e}^3cos^3\varphi)(1 - \dot{e}cos\varphi)^{-2}\, d\varphi +$$

$$+ \dot{e}^{-3}\int_0^{2\pi}(1 - 3\dot{e}cos\varphi + 3\dot{e}^2cos^2\varphi)(1 - \dot{e}cos\varphi)^{-2}\, d\varphi = - \dot{e}^{-3}\int_0^{2\pi}(1 - \dot{e}cos\varphi)\, d\varphi +$$

$$+ \dot{e}^{-3}\, \mathbf{I_0}(0,\dot{e},1) - (3\dot{e}/\dot{e}^3)\mathbf{I_1}(0,\dot{e},1) + (3\dot{e}^2/\dot{e}^3)\mathbf{I_2}(0,\dot{e},1) =$$
$$= - 2\pi\dot{e}^{-3} + 0 + \dot{e}^{-3}\, \mathbf{I_0}(0,\dot{e},1) - 3\dot{e}^{-2}\, \mathbf{I_1}(0,\dot{e},1) + 3\dot{e}^{-1}\, \mathbf{I_2}(0,\dot{e},1) .$$

Let us evaluate the third nonzero term in the right-hand side:

(26) $$- 3\dot{e}^{-2}\, \mathbf{I_1}(0,\dot{e},1) = - 3(2\dot{e}^3)^{-1}\int_0^{2\pi}2\dot{e}cos\varphi(1 - \dot{e}cos\varphi)^{-2}\, d\varphi =$$

$$= 3(2\dot{e}^3)^{-1}\int_0^{2\pi}(1 - 2\dot{e}cos\varphi + \dot{e}^2cos^2\varphi)(1 - \dot{e}cos\varphi)^{-2}\, d\varphi -$$

$$- 3(2\dot{e}^3)^{-1}\int_0^{2\pi}(1 + \dot{e}^2cos^2\varphi)(1 - \dot{e}cos\varphi)^{-2}\, d\varphi =$$

$$= 3(2\dot{e}^3)^{-1}\, 2\pi - 3(2\dot{e}^3)^{-1}\mathbf{I_0}(0,\dot{e},1) - 3(2\dot{e})^{-1}\mathbf{I_2}(0,\dot{e},1) .$$

Substituting this result into equation (25), we obtain:

(27) $$\mathbf{I_3}(0,\dot{e},1) = [- \dot{e}^{-3} + 3(2\dot{e}^3)^{-1}]2\pi + [\dot{e}^{-3} - 3(2\dot{e}^3)^{-1}]\mathbf{I_0}(0,\dot{e},1) +$$
$$+ [3\dot{e}^{-1} - 3(2\dot{e})^{-1}\mathbf{I_2}(0,\dot{e},1) ,$$ or

(28) $$\mathbf{I_3}(0,\dot{e},1) = 2\pi\, (2\dot{e}^{-3}) - (2\dot{e}^{-3})\mathbf{I_0}(0,\dot{e},1) + 3(2\dot{e})^{-1}\mathbf{I_2}(0,\dot{e},1) .$$

In straightforward way we find that:

(29) $$3\dot{e}^{-1}\mathbf{I_2}(0,\dot{e},1) = 3\dot{e}^{-1}\int_0^{2\pi}cos^2\varphi(1 - \dot{e}cos\varphi)^{-2}\, d\varphi = 3\dot{e}^{-3}\int_0^{2\pi}\dot{e}^2cos^2\varphi(1 - \dot{e}cos\varphi)^{-2}\, d\varphi =$$

$$= 3\dot{e}^{-3}\int_0^{2\pi}(1 - 2\dot{e}cos\varphi + \dot{e}^2cos^2\varphi)(1 - \dot{e}cos\varphi)^{-2}\, d\varphi - 3\dot{e}^{-3}\int_0^{2\pi}(1 - 2\dot{e}cos\varphi) \times$$
$$\times (1 - \dot{e}cos\varphi)^{-2}\, d\varphi = 3\dot{e}^{-3}2\pi - 3\dot{e}^{-3}\mathbf{I_0}(0,\dot{e},1) + 6\dot{e}^{-2}\mathbf{I_1}(0,\dot{e},1) .$$

Dividing the both sides of this equality by 6, we have:

(30) $$(2\dot{e}^3)^{-1}2\pi = (2\dot{e}^3)^{-1}\mathbf{I_0}(0,\dot{e},1) - \dot{e}^{-2}\mathbf{I_1}(0,\dot{e},1) + (2\dot{e})^{-1}\mathbf{I_2}(0,\dot{e},1) .$$

Substituting this into (28) and multiplying by $\dot{e}$, we shall finally obtain the following *intermediate result*:

(31) $$0 = \dot{e}^{-1}\mathbf{I_1}(0,\dot{e},1) - 2\mathbf{I_2}(0,\dot{e},1) + \dot{e}\mathbf{I_3}(0,\dot{e},1)\ .$$

Our further computations include explicit analytical evaluations of the integrals $\mathbf{I_0}(0,\dot{e},1)$ and $\mathbf{I_3}(0,\dot{e},1)$:

(32) $$\mathbf{I_0}(0,\dot{e},1) = \int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-2}\, d\varphi = 2\pi\,(1 - \dot{e}^2)^{-3/2}\ ,$$

according to formula **858.535** from [14].

(33) $$\mathbf{I_1}(0,\dot{e},1) = \int_0^{2\pi}cos\varphi(1 - \dot{e}cos\varphi)^{-2}\, d\varphi = -\,\dot{e}^{-1}\int_0^{2\pi}[(1 - \dot{e}cos\varphi) - 1](1 - \dot{e}cos\varphi)^{-2}\, d\varphi\ =$$

$$= \dot{e}^{-1}\int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-2}\, d\varphi - \dot{e}^{-1}\int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-1}\, d\varphi\ .$$

From Dwight [14], formula **858.525**, we find that:

(34) $$\int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-1}\, d\varphi = 2\pi\,(1 - \dot{e}^2)^{-1/2}\ .$$

Combining evaluations (32) and (34) into (33), the result is:

(35) $$\mathbf{I_1}(0,\dot{e},1) = 2\pi\,\dot{e}^{-1}(1 - \dot{e}^2)^{-1}(1 - \dot{e}^2)^{-1/2}\ - 2\pi\,\dot{e}^{-1}(1 - \dot{e}^2)^{-1/2}\ =$$
$$=\ 2\pi\,\dot{e}(1 - \dot{e}^2)^{-3/2}\ = \dot{e}\mathbf{I_0}(0,\dot{e},1)\ .$$

Substituting the above result into (31), we finally obtain the necessary relation (24).

3.1.3. <u>Case $n \neq 0$, $e(u) \neq 0$, $e(u) - \dot{e}(u) = 0$ => $\dot{e}(u) = e(u) \neq 0$.</u>

The relation (16) can be written as:

(36) $$0 = \mathbf{I_0}(e,\dot{e} = e,n) + n\dot{e}\mathbf{I_1}(e,\dot{e} = e,n) - 2\mathbf{I_2}(e,\dot{e} = e,n) - (n + 1)\dot{e}\mathbf{I_3}(e,\dot{e} = e,n)\ .$$

We compute directly that:

(37) $$\mathbf{I_3}(e,\dot{e} = e,n) = \int_0^{2\pi}cos^3\varphi(1 + ecos\varphi)^{n-2}\, d\varphi = e^{-1}\int_0^{2\pi}[(1 + ecos\varphi) - 1](1 + ecos\varphi)^{n-2}\ \times$$

$$\times\ cos^2\varphi\ d\varphi = e^{-1}\int_0^{2\pi}cos^2\varphi(1 + ecos\varphi)^{n-1}d\varphi\ - e^{-1}\int_0^{2\pi}cos^2\varphi(1 + ecos\varphi)^{n-2}\, d\varphi =$$

$$=\ -\,e^{-1}\mathbf{I_2}(e,\dot{e} = e,n) + e^{-1}\int_0^{2\pi}cos\varphi(1 + ecos\varphi)^{n-1}d\ sin\varphi = -\,e^{-1}\mathbf{I_2}(e,\dot{e} = e,n)\ +$$

$$+\ e^{-1}\{\ sin\varphi\ cos\varphi(1 + ecos\varphi)^{n-1}\Big|_0^{2\pi} - \int_0^{2\pi}sin\varphi\ d[cos\varphi(1 + ecos\varphi)^{n-1}]\ \}\ =$$

$$= - e^{-1}\mathbf{I_2}(e,\dot{e} = e,n) + e^{-1}\int_0^{2\pi} sin^2\varphi(1 + ecos\varphi)^{n-1}d\varphi + (n-1)e^{-1}\int_0^{2\pi} sin\varphi\ cos\varphi\ \times$$

$$\times (1 + ecos\varphi)^{n-2}esin\varphi\ d\varphi = - e^{-1}\mathbf{I_2}(e,\dot{e} = e,n) + e^{-1}\int_0^{2\pi}(1 - cos^2\varphi)(1 + ecos\varphi)^{n-1}d\varphi +$$

$$+ (n-1)\int_0^{2\pi} cos\varphi\ sin^2\varphi(1 + ecos\varphi)^{n-2}d\varphi = - e^{-1}\mathbf{I_2}(e,\dot{e} = e,n) + e^{-1}\int_0^{2\pi}(1 + ecos\varphi)^{n-2}d\varphi +$$

$$+ \int_0^{2\pi} cos\varphi(1 + ecos\varphi)^{n-2}d\varphi - e^{-1}\int_0^{2\pi} cos^2\varphi(1 + ecos\varphi)^{n-2}d\varphi - \int_0^{2\pi} cos^3\varphi(1 + ecos\varphi)^{n-2}d\varphi +$$

$$+ (n-1)\int_0^{2\pi} cos\varphi(1 + ecos\varphi)^{n-2}d\varphi - (n-1)\int_0^{2\pi} cos^3\varphi\ (1 + ecos\varphi)^{n-2}d\varphi\ =$$

$$= - e^{-1}\mathbf{I_2}(e,\dot{e} = e,n) + e^{-1}\mathbf{I_0}(e,\dot{e} = e,n) + \mathbf{I_1}(e,\dot{e} = e,n) - e^{-1}\mathbf{I_2}(e,\dot{e} = e,n) -$$
$$- \mathbf{I_3}(e,\dot{e} = e,n) + (n-1)\mathbf{I_1}(e,\dot{e} = e,n) - (n-1)\mathbf{I_3}(e,\dot{e} = e,n)\ .$$

Therefore:

(38) $$0 = e^{-1}\mathbf{I_0}(e,\dot{e} = e,n) + n\mathbf{I_1}(e,\dot{e} = e,n) - 2e^{-1}\mathbf{I_2}(e,\dot{e} = e,n) - (n+1)\mathbf{I_3}(e,\dot{e} = e,n)\ .$$

Multiplying (38) by e and taking into account that for the considered value of u $e(u) = \dot{e}(u)$, we complete the proof of the linear relation (36).

3.1.4. Case $n = 0$, $e(u) = 0$, $e(u) - \dot{e}(u) = 0$ => $\dot{e}(u) = 0$.

The relation (16) can be written as:

(39) $$0 = \mathbf{I_0}(0,0,0) - 2\mathbf{I_2}(0,0,0)\ .$$

In this case $\mathbf{I_0}(0,0,0) = \int_0^{2\pi} d\varphi = 2\pi$ and $\mathbf{I_2}(0,0,0) = \int_0^{2\pi} cos^2\varphi\ d\varphi = \pi$. Then (39) immediately follows.

3.1.5. Case $n = 0$, $e(u) = 0$, $e(u) - \dot{e}(u) \neq 0$ => $\dot{e}(u) \neq 0$.

The relation (16) becomes:

(40) $$0 = \mathbf{I_0}(0,\dot{e},0) - \dot{e}\mathbf{I_1}(0,\dot{e},0) - 2\mathbf{I_2}(0,\dot{e},0) + 2\dot{e}\mathbf{I_3}(0,\dot{e},0)\ .$$

The direct computation gives:

(41) $$\mathbf{I_3}(0,\dot{e},0) = \int_0^{2\pi} cos^3\varphi(1 - \dot{e}cos\varphi)^{-1}d\varphi = - \dot{e}^{-1}\int_0^{2\pi} cos^2\varphi[(1 - \dot{e}cos\varphi) - 1](1 - \dot{e}cos\varphi)^{-1}d\varphi =$$

$$= \dot{e}^{-1}\int_0^{2\pi} cos^2\varphi(1 - \dot{e}cos\varphi)^{-1}d\varphi - \dot{e}^{-1}\int_0^{2\pi} cos^2\varphi\ d\varphi = \dot{e}^{-1}\mathbf{I_2}(0,\dot{e},0) - \pi\ \dot{e}^{-1}\ .$$

Multiplying by $2\dot{e}$, we shall obtain:

(42) $2\dot{e}\mathbf{I}_3(0,\dot{e},0) - 2\mathbf{I}_2(0,\dot{e},0) = -2\pi$.

Further we also evaluate that:

(43) $\mathbf{I}_1(0,\dot{e},0) = \int_0^{2\pi} cos\varphi(1 - \dot{e}cos\varphi)^{-1}d\varphi = -\dot{e}^{-1}\int_0^{2\pi}[(1 - \dot{e}cos\varphi) - 1](1 - \dot{e}cos\varphi)^{-1}d\varphi =$

$= -2\pi\dot{e}^{-1} + \dot{e}^{-1}\int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-1}d\varphi$.

Consequently:

(44) $\dot{e}\mathbf{I}_1(0,\dot{e},0) - \mathbf{I}_0(0,\dot{e},0) = -2\pi$.

Combining (42) and (44), we attain to the relation (40).

3.1.6. <u>Case $n = 0$, $e(u) \neq 0$, $e(u) - \dot{e}(u) = 0$ => $\dot{e}(u) = e(u) \neq 0$.</u>

The linear relation (16) can be written as:

(45) $0 = \mathbf{I}_0(e,\dot{e} = e,0) - 2\mathbf{I}_2(e,\dot{e} = e,0) - 3\dot{e}\mathbf{I}_3(e,\dot{e} = e,0)$.

To prove the above statement, we must perform evaluation of the integrals

$\mathbf{I}_0(e,\dot{e} = e,0) = \int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-2}d\varphi$, …, $\mathbf{I}_3(e,\dot{e} = e,0) = \int_0^{2\pi}cos^3\varphi(1 - \dot{e}cos\varphi)^{-2}d\varphi$. Clearly, this is fully analogous to the estimation of the integrals in the case **3.1.2.2.** We must only replace $-\dot{e}(u)$ in the denominators of the integrals by $e(u)$ and proceed by the same way, when we were proving the relation (24). We shall not write out these clumsy calculations again, in order to prove validity of the relation (45).

3.1.7. <u>Case $n = 0$, $e(u) \neq 0$, $e(u) - \dot{e}(u) \neq 0$.</u>

The linear relation (16) now becomes:

(46) $e(e - \dot{e})\mathbf{I}_4(e,\dot{e},0) = \mathbf{I}_0(e,\dot{e},0) + (e - \dot{e})\mathbf{I}_1(e,\dot{e},0) - 2\mathbf{I}_2(e,\dot{e},0) - (3e - 2\dot{e})\mathbf{I}_3(e,\dot{e},0)$.

In an earlier paper [11] (formulas (3a) – (3d)) we have already derived in explicit form ***analytical*** expressions for the integrals, entering in (46). We shall now rewrite in a little more compact form these results. Let us denote by $\boldsymbol{A}(e,\dot{e})$ the multiplier:

(47) $\boldsymbol{A}(e,\dot{e}) = 2\pi\dot{e}^{-2}(1 - e^2)^{-3/2}[1 - (e - \dot{e})^2]^{-1/2}$.

Then, according to [11] (formulas (3a) – (3d)), with this simplification of the notations, we have:

(48) $\mathbf{I}_0(e,\dot{e},0) = \boldsymbol{A}(e,\dot{e})\{ e\dot{e}[1 - (e - \dot{e})^2]^{1/2} - e(e - \dot{e})(1 - e^2)[1 - (e - \dot{e})^2]^{1/2} +$
$+ (e - \dot{e})^2(1 - e^2)^{3/2} \}$,

$$(49)\qquad \mathbf{I}_1(e,\dot{e},0) = \boldsymbol{A}(e,\dot{e})\{ (e - \dot{e} - e^3)[1 - (e - \dot{e})^2]^{1/2} - (e - \dot{e})(1 - e^2)^{3/2} \},$$

$$(50)\qquad \mathbf{I}_2(e,\dot{e},0) = \boldsymbol{A}(e,\dot{e})\{ (-1 + e^2 + e\dot{e})[1 - (e - \dot{e})^2]^{1/2} + (1 - e^2)^{3/2} \},$$

$$(51)\qquad \mathbf{I}_3(e,\dot{e},0) = \boldsymbol{A}(e,\dot{e})e^{-2}(e - \dot{e})^{-1}\{- e^2(1 - e^2)^{3/2} + [e^2 - e^4 - e^3\dot{e} - \dot{e}^2 + 2e^2\dot{e}^2 + \dot{e}^2(1 - e^2)^{3/2}][1 - (e - \dot{e})^2]^{1/2} \},$$

$$(52)\qquad e(e - \dot{e})\mathbf{I}_4(e,\dot{e},0) = \boldsymbol{A}(e,\dot{e})e^{-2}(e - \dot{e})^{-1}\{ e^3(1 - e^2)^{3/2} + (- e^3 + e^5 + e^4\dot{e} + 3e\dot{e}^2 - 5e^3\dot{e}^2 - 2\dot{e}^3 + 3e^2\dot{e}^3)[1 - (e - \dot{e})^2]^{1/2} + (- 3e\dot{e}^2 + 2\dot{e}^3)(1 - e^2)^{3/2}[1 - (e - \dot{e})^2]^{1/2} \}.$$

Let us now compute the right-hand side of the equality (46):

$$(53)\qquad \mathbf{I}_0(e,\dot{e},0) + (e - \dot{e})\mathbf{I}_1(e,\dot{e},0) - 2\mathbf{I}_2(e,\dot{e},0) - (3e - 2\dot{e})\mathbf{I}_3(e,\dot{e},0) = \boldsymbol{A}(e,\dot{e})e^{-2}(e - \dot{e})^{-1} \times \\ \times \{ e^2(e - \dot{e})e\dot{e}[1 - (e - \dot{e})^2]^{1/2} - e^3(e - \dot{e})^2(1 - e^2)[1 - (e - \dot{e})^2]^{1/2} + \\ + e^2(e - \dot{e})^3(1 - e^2)^{3/2} + e^2(e - \dot{e})^2(e - \dot{e} - e^3)[1 - (e - \dot{e})^2]^{1/2} - e^2(e - \dot{e})^2(1 - e^2)^{3/2} + \\ + (2 - 2e^2 - 2e\dot{e})e^2(e - \dot{e})[1 - (e - \dot{e})^2]^{1/2} - 2e^2(e - \dot{e})(1 - e^2)^{3/2} + \\ + (3e - 2\dot{e})e^2(1 - e^2)^{3/2} + (- 3e + 2\dot{e})(e^2 - e^4 - e^3\dot{e} - \dot{e}^2 + 2e^2\dot{e}^2 + \\ + \dot{e}^2(1 - e^2)^{3/2}[1 - (e - \dot{e})^2]^{1/2} \} = \\ = \boldsymbol{A}(e,\dot{e})e^{-2}(e - \dot{e})^{-1}\{(- e^3 + e^5 + e^4\dot{e} + 3e\dot{e}^2 - 5e^3\dot{e}^2 - 2\dot{e}^3 + 3e^2\dot{e}^3)[1 - (e - \dot{e})^2]^{1/2} + \\ + e^3(1 - e^2)^{3/2} + (- 3e\dot{e}^2 + 2\dot{e}^3)(1 - e^2)^{3/2}[1 - (e - \dot{e})^2]^{1/2} \} = e(e - \dot{e})\mathbf{I}_4(e,\dot{e},0) .$$

Hence, the linear relation (46) is proved. With this, we have also completed the validity of relation (16) in the general case. That is, for *integer*/*noninteger* powers n, *zero*/*nonzero* values of $e(u)$ (for $|e(u)| < 1$) and $\dot{e}(u)$ (for $|\dot{e}(u)| < 1$), and also for *zero*/*nonzero* values of $[e(u) - \dot{e}(u)]$ (for $|e(u) - \dot{e}(u)| < 1$).

3.2. <u>Elimination of the integral $\mathbf{I}_2(e,\dot{e},n)$</u>

Generally speaking, the approach in the computing of $\mathbf{I}_4(e,\dot{e},n)$ is the following: we perform a series of evaluations of $\mathbf{I}_4(e,\dot{e},n)$, decomposing its integrand into such, containing into their nominators powers of $cos\varphi$ equal or less than 4, and the same denominators $[1 + (e - \dot{e})cos\varphi]^{n+1}$. After that, we transfer the repeatedly appeared integrals $\mathbf{I}_4(e,\dot{e},n)$ in the *right-hand* side into the *left-hand* side, in order to combine all $\mathbf{I}_4(e,\dot{e},n)$. Unfortunately, such a procedure ***does not work at all*** when we try to apply it for elimination of the integral $\mathbf{I}_3(e,\dot{e},n)$ (we suppose that the linear relation (16) is already used for removing of the integral $\mathbf{I}_4(e,\dot{e},n)$). The reason for this unsuccessful attempt is that the multiplier before $\mathbf{I}_3(e,\dot{e},n)$ equals to zero for all values of the variable u. It is suspected that this impossibility is in relation to the linear independence of the considered seven integrals (1), (2) and (3). We shall not deal with this problem in the present paper and continue to the evaluation of the integral $\mathbf{I}_2(e,\dot{e},n)$.

According to the definition (3), we can write:

$$(54)\qquad \mathbf{I}_3(e,\dot{e},n) = \int_0^{2\pi} cos^3\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}\, d\varphi \ =$$

$$= e^{-1}\int_0^{2\pi} cos^2\varphi[(1 + ecos\varphi) - 1](1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}\, d\varphi =$$

$$= e^{-1}\int_0^{2\pi} cos^2\varphi(1 + ecos\varphi)^{n-1}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}\, d\varphi -$$

$$- e^{-1}\int_0^{2\pi} cos^2\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}\, d\varphi =$$

$$= - e^{-1}\mathbf{I_2}(e,\dot{e},n) + e^{-1}\int_0^{2\pi}(1 - sin^2\varphi)(1 + ecos\varphi)^{n-1}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}\, d\varphi =$$

$$= - e^{-1}\mathbf{I_2}(e,\dot{e},n) + e^{-1}\int_0^{2\pi}(1 + ecos\varphi)^{n-1}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}\, d\varphi +$$

$$+ [e(e - \dot{e})]^{-1}\int_0^{2\pi} sin\varphi(1 + ecos\varphi)^{n-1}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}\, d[1 + (e - \dot{e})cos\varphi] .$$

Applying a relation analogous to (7), we obtain:

(55) $$\mathbf{I_3}(e,\dot{e},n) = - e^{-1}\mathbf{I_2}(e,\dot{e},n) + e^{-1}\mathbf{I_0}(e,\dot{e},n) + \mathbf{I_1}(e,\dot{e},n) -$$

$$- [ne(e - \dot{e})]^{-1}\int_0^{2\pi} sin\varphi(1 + ecos\varphi)^{n-1} d\{ [1 + (e - \dot{e})cos\varphi]^{-n} \} =$$

$$= e^{-1}\mathbf{I_0}(e,\dot{e},n) + \mathbf{I_1}(e,\dot{e},n) - e^{-1}\mathbf{I_2}(e,\dot{e},n) - [ne(e - \dot{e})]^{-1}\{ sin\varphi(1 + ecos\varphi)^{n-1} \times$$

$$\times [1 + (e - \dot{e})cos\varphi]^{-n} \Big|_0^{2\pi} \} + [ne(e - \dot{e})]^{-1}\{ \int_0^{2\pi} cos\varphi(1 + ecos\varphi)^{n-1}[1 + (e - \dot{e})cos\varphi]^{-n} d\varphi -$$

$$- e(n - 1)\int_0^{2\pi} sin^2\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-n} d\varphi \} =$$

$$= e^{-1}\mathbf{I_0}(e,\dot{e},n) + \mathbf{I_1}(e,\dot{e},n) - e^{-1}\mathbf{I_2}(e,\dot{e},n) + [ne(e - \dot{e})]^{-1}\{ \int_0^{2\pi} cos\varphi[1 + (e - \dot{e})cos\varphi] \times$$

$$\times(1 + ecos\varphi)^{n-1}[1 + (e - \dot{e})cos\varphi]^{-(n+1)} d\varphi - (n - 1)e\{ \int_0^{2\pi}(1 + ecos\varphi)^{n-2} \times$$

$$\times [1 + (e - \dot{e})cos\varphi]^{-(n+1)} d\varphi + (e - \dot{e})\int_0^{2\pi} cos\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-(n+1)} d\varphi -$$

$$- \int_0^{2\pi} cos^2\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-(n+1)} d\varphi -$$

$$- (e - \dot{e}) \int_0^{2\pi} cos^3\varphi(1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}d\varphi \} \} =$$

$$= e^{-1}\mathbf{I_0}(e,\dot{e},n) + \mathbf{I_1}(e,\dot{e},n) - e^{-1}\mathbf{I_2}(e,\dot{e},n) + [ne(e - \dot{e})]^{-1}[\mathbf{I_1}(e,\dot{e},n) + e\mathbf{I_2}(e,\dot{e},n) + (e - \dot{e})\mathbf{I_2}(e,\dot{e},n) + e(e - \dot{e})\mathbf{I_3}(e,\dot{e},n) - (n-1)e\mathbf{I_0}(e,\dot{e},n) - (n-1)e(e - \dot{e})\mathbf{I_1}(e,\dot{e},n) + (n-1)e\mathbf{I_2}(e,\dot{e},n) + (n-1)e(e - \dot{e})\mathbf{I_3}(e,\dot{e},n)] .$$

Consequently, we have the following expression:

(56) $$\mathbf{I_3}(e,\dot{e},n) = \{e^{-1} - (n-1)[n(e - \dot{e})]^{-1}\}\mathbf{I_0}(e,\dot{e},n) + \{1 + [ne(e - \dot{e})]^{-1} - (n-1)n^{-1}\}\mathbf{I_1}(e,\dot{e},n) + \{- e^{-1} + [n(e - \dot{e})]^{-1} + (ne)^{-1} + (n-1)[n(e - \dot{e})]^{-1}\}\mathbf{I_2}(e,\dot{e},n) + [n^{-1} + (n-1)n^{-1}]\mathbf{I_3}(e,\dot{e},n) .$$

It is evident that the integrals $\mathbf{I_3}(e,\dot{e},n)$ from the both sides of the above equality cancel out, and it is impossible to determine any linear relation between $\mathbf{I_3}(e,\dot{e},n)$, $\mathbf{I_0}(e,\dot{e},n)$, $\mathbf{I_1}(e,\dot{e},n)$, and $\mathbf{I_2}(e,\dot{e},n)$, ***as already mentioned above***. Nevertheless, the result (56) may be used to eliminate the integral $\mathbf{I_2}(e,\dot{e},n)$. Multiplying (56) by $ne(e - \dot{e})$, we obtain:

(57) $$[e + (n-1)\dot{e}]\mathbf{I_2}(e,\dot{e},n) = (- e + n\dot{e})\mathbf{I_0}(e,\dot{e},n) - [1 + e(e - \dot{e})]\mathbf{I_1}(e,\dot{e},n) .$$

In the above derivation, we again have supposed that simultaneously are fulfilled the following three conditions: $n \neq 0$, $e(u) \neq 0$, $e(u) - \dot{e}(u) \neq 0$, for every considered value of the independent variable $u \equiv ln\ p$ (p is the focal parameter of the particle orbit). We note that (57) makes sense even if some (or even all) of these restrictions are violated.

3.2.1. Case $n \neq 0$, $e(u) = 0$, $e(u) - \dot{e}(u) = 0$ => $e(u) = \dot{e}(u) = 0$.

The relation (57) is obviously satisfied, because $\mathbf{I_1}(0,0,n) = \int_0^{2\pi} cos\varphi\, d\varphi = 0$.

3.2.2. Case $n \neq 0$, $e(u) = 0$, $e(u) - \dot{e}(u) \neq 0$ => $\dot{e}(u) \neq 0$.

The relation (57) takes the form:

(58) $$(n-1)\dot{e}\mathbf{I_2}(0,\dot{e},n) = n\dot{e}\mathbf{I_0}(0,\dot{e},n) - \mathbf{I_1}(0,\dot{e},n) .$$

We compute that:

(59) $$\mathbf{I_2}(0,\dot{e},n) = \int_0^{2\pi} \cos^2\varphi(1 - \dot{e}cos\varphi)^{-(n+1)}d\varphi = \int_0^{2\pi}(1 - \dot{e}cos\varphi)^{-(n+1)}d\varphi -$$

$$- \int_0^{2\pi} sin^2\varphi(1 - \dot{e}cos\varphi)^{-(n+1)}d\varphi = \mathbf{I_0}(0,\dot{e},n) - \dot{e}^{-1}\int_0^{2\pi} sin\varphi(1 - \dot{e}cos\varphi)^{-(n+1)}d(1 - \dot{e}cos\varphi) =$$

$$= \mathbf{I_0}(0,\dot{e},n) + (n\dot{e})^{-1}\int_0^{2\pi} sin\varphi d[(1 - \dot{e}cos\varphi)^{-n}] = \mathbf{I_0}(0,\dot{e},n) + (n\dot{e})^{-1}\{sin\varphi(1 - \dot{e}cos\varphi)^{-n}\Big|_0^{2\pi} -$$

$$- \int_0^{2\pi} cos\varphi(1 - \dot{e}cos\varphi)^{-n}d\varphi\} = \mathbf{I_0}(0,\dot{e},n) - (n\dot{e})^{-1}\int_0^{2\pi} cos\varphi(1 - \dot{e}cos\varphi)(1 - \dot{e}cos\varphi)^{-(n+1)}d\varphi =$$

$= \mathbf{I_0}(0,\dot{e},n) - (n\dot{e})^{-1}\mathbf{I_1}(0,\dot{e},n) + n^{-1}\mathbf{I_2}(0,\dot{e},n)$.

From this equality follows:

(60) $(1 - n^{-1})\mathbf{I_2}(0,\dot{e},n) \equiv (n-1)\dot{e}(n\dot{e})^{-1}\mathbf{I_2}(0,\dot{e},n) = \mathbf{I_0}(0,\dot{e},n) - (n\dot{e})^{-1}\mathbf{I_1}(0,\dot{e},n)$.

Multiplying by $n\dot{e} \neq 0$, we obtain the seeked equality (58).

3.2.3. Case $n \neq 0$, $e(u) \neq 0$, $e(u) - \dot{e}(u) = 0$ => $\dot{e}(u) = e(u) \neq 0$.

The relation (57) in this case becomes:

(61) $[e + (n-1)\dot{e}]\mathbf{I_2}(e,\dot{e} = e,n) \equiv n\dot{e}\mathbf{I_2}(e,\dot{e} = e,n) = (n-1)\dot{e}\mathbf{I_0}(e,\dot{e} = e,n) - \mathbf{I_1}(e,\dot{e} = e,n)$.

We directly compute that:

(62) $$\mathbf{I_2}(e,\dot{e} = e,n) = \int_0^{2\pi} cos^2\varphi(1 + ecos\varphi)^{n-2}d\varphi = e^{-1}\int_0^{2\pi} cos\varphi[(1 + ecos\varphi) - 1] \times$$

$$\times(1 + ecos\varphi)^{n-2}d\varphi = - e^{-1}\int_0^{2\pi} cos\varphi(1 + ecos\varphi)^{n-2}d\varphi + e^{-1}\int_0^{2\pi}(1 + ecos\varphi)^{n-1}d\, sin\varphi =$$

$$= - e^{-1}\mathbf{I_1}(e,\dot{e} = e,n) + e^{-1}\{ sin\varphi(1 + ecos\varphi)^{n-1} \Big|_0^{2\pi} - \int_0^{2\pi} sin\varphi\, d[(1 + ecos\varphi)^{n-1}]\} =$$

$$= - e^{-1}\mathbf{I_1}(e,\dot{e} = e,n) + (n-1)\int_0^{2\pi}(1 - cos^2\varphi)(1 + ecos\varphi)^{n-2}d\varphi =$$

$$= - e^{-1}\mathbf{I_1}(e,\dot{e} = e,n) + (n-1)\mathbf{I_0}(e,\dot{e} = e,n) - (n-1)\mathbf{I_2}(e,\dot{e} = e,n) .$$

Multiplication of the both sides by $\dot{e}(u) = e(u) \neq 0$ gives the result:

(63) $n\dot{e}\mathbf{I_2}(e,\dot{e} = e,n) = (n-1)\dot{e}\mathbf{I_0}(e,\dot{e} = e,n) - \mathbf{I_1}(e,\dot{e} = e,n)$,

that proves (61).

3.2.4. Case $n = 0$, $e(u) = 0$, $e(u) - \dot{e}(u) = 0$ => $\dot{e}(u) = 0$.

The relation (57) is obviously true, because $\mathbf{I_1}(0,0,0) = \int_0^{2\pi} cos\varphi\, d\varphi = 0$.

3.2.5. Case $n = 0$, $e(u) = 0$, $e(u) - \dot{e}(u) \neq 0$ => $\dot{e}(u) \neq 0$.

The relation (57) now becomes:

(64) $- \dot{e}\mathbf{I_2}(0,\dot{e},0) = - \mathbf{I_1}(0,\dot{e},0)$.

It is evident that:

(65) $$\mathbf{I_2}(0,\dot{e},0) = \int_0^{2\pi} cos^2\varphi(1 - \dot{e}cos\varphi)^{-1}d\varphi = - \dot{e}^{-1}\int_0^{2\pi} cos\varphi[(1 - \dot{e}cos\varphi) - 1](1 - \dot{e}cos\varphi)^{-1}d\varphi =$$

$$= - \dot{e}^{-1}\int_0^{2\pi} cos\varphi\, d\varphi + \dot{e}^{-1}\int_0^{2\pi} cos\varphi(1 - \dot{e}cos\varphi)^{-1}d\varphi = \dot{e}^{-1}\mathbf{I_1}(0,\dot{e},0) .$$

Multiplication of the above equality by $-\dot{e}(u) \neq 0$ gives (64).

3.2.6. <u>Case $n = 0, e(u) \neq 0, e(u) - \dot{e}(u) = 0 \Rightarrow e(u) = \dot{e}(u) \neq 0$.</u>

The relation (57) can be written in the following form:

(66) $$0 = -e\mathbf{I_0}(e,\dot{e} = e,0) - \mathbf{I_1}(e,\dot{e} = e,0) .$$

We directly compute that:

(67) $$\mathbf{I_1}(e,\dot{e} = e,0) = \int_0^{2\pi} cos\varphi(1 + ecos\varphi)^{-2}d\varphi = e^{-1}\int_0^{2\pi}[(1 + ecos\varphi) - 1](1 + ecos\varphi)^{-2}d\varphi =$$

$$= -e^{-1}\int_0^{2\pi}(1 + ecos\varphi)^{-2}d\varphi + e^{-1}\int_0^{2\pi}(1 + ecos\varphi)d\varphi =$$

$$= -e^{-1}\mathbf{I_0}(e,\dot{e} = e,0) + 2\pi\, e^{-1}(1 - e^2)^{-1/2},$$

where we have used formula **858.525** from [14]. But according to formula **858.535** from the same source [14]:

(68) $$\mathbf{I_0}(e,\dot{e} = e,0) = \int_0^{2\pi}(1 + ecos\varphi)^{-2}d\varphi = 2\pi\,(1 - e^2)^{-3/2} = (1 - e^2)^{-1}\,2\pi\,(1 - e^2)^{-1/2},$$

which means that:

(69) $$2\pi\,(1 - e^2)^{-1/2} = (1 - e^2)\mathbf{I_0}(e,\dot{e} = e,0) .$$

Substituting this result into (67), we have that:

(70) $$\mathbf{I_1}(e,\dot{e} = e,0) = -e^{-1}\mathbf{I_0}(e,\dot{e} = e,0) + e^{-1}(1 - e^2)\mathbf{I_0}(e,\dot{e} = e,0) = -e\,\mathbf{I_0}(e,\dot{e} = e,0) .$$

Hence, (66) is proved.

3.2.7. <u>Case $n = 0, e(u) \neq 0, e(u) - \dot{e}(u) \neq 0$.</u>

The linear relation (57) can be written as:

(71) $$(e - \dot{e})\mathbf{I_2}(e,\dot{e},0) = -e\mathbf{I_0}(e,\dot{e},0) - [1 + e(e - \dot{e})]\mathbf{I_1}(e,\dot{e},0) .$$

Let us calculate at first, using as before, the explicit expressions for the integrals from paper [11] (formulas (3a) – (3c)). In the present paper we have written them as the expressions (48), (49) and (50) for $\mathbf{I_0}(e,\dot{e},0)$, $\mathbf{I_1}(e,\dot{e},0)$, and $\mathbf{I_2}(e,\dot{e},0)$, respectively. From (50) we obtain:

(72) $$(e - \dot{e})\mathbf{I_2}(e,\dot{e},0) = \boldsymbol{A}(e,\dot{e})\{(e - \dot{e})(-1 + e^2 + e\dot{e})[1 - (e - \dot{e})^2]^{1/2} + (e - \dot{e})(1 - e^2)^{3/2}\}.$$

Further we evaluate the right-hand side of (71):

(73) $$-e\,\mathbf{I_0}(e,\dot{e},0) - [1 + e(e - \dot{e})]\mathbf{I_1}(e,\dot{e},0) = \boldsymbol{A}(e,\dot{e})\{-e^2\dot{e}[1 - (e - \dot{e})^2]^{1/2} +$$
$$+ e^2(e - \dot{e})(1 - e^2)[1 - (e - \dot{e})^2]^{1/2} - e(e - \dot{e})^2(1 - e^2)^{3/2} - [1 + e(e - \dot{e})](e - \dot{e} - e^3) \times$$
$$\times [1 - (e - \dot{e})^2]^{1/2} + [1 + e(e - \dot{e})](e - \dot{e})\,(1 - e^2)^{3/2}\} =$$
$$= \boldsymbol{A}(e,\dot{e})\{(-e + e^3 + \dot{e} - e\dot{e}^2)[1 - (e - \dot{e})^2]^{1/2} + (e - \dot{e})(1 - e^2)^{3/2}\} .$$

But $(e - \dot{e})(-1 + e^2 + e\dot{e}) = -e + e^3 + \dot{e} - e\dot{e}^2$ and hence:

(74) $$-e\,\mathbf{I_0}(e,\dot{e},0) - [1 + e(e - \dot{e})]\mathbf{I_1}(e,\dot{e},0) =$$
$$= \boldsymbol{A}(e,\dot{e})\{(e - \dot{e})(-1 + e^2 + e\dot{e})[1 - (e - \dot{e})^2]^{1/2} + (e - \dot{e})(1 - e^2)^{3/2}\}.$$

The right-hand sides of (72) and (74) coincide and this proves the relation (71). This also completes the validity of (57)in the general case of *integer*/*noninteger* powers n, *zero*/*nonzero* values of $e(u)$ (for $|e(u)| < 1$) and *zero*/*nonzero* values of $[e(u) - \dot{e}(u)]$ (for $|e(u) - \dot{e}(u)| < 1$).

3.3. Elimination of the integral $\mathbf{I_1}(e,\dot{e},n)$

In the next derivation we shall use not only definitions (3), but also definition (1) and (2). According to the later and the identity $cos\varphi^2 + sin\varphi^2 = 1$, we have:

(75) $$\mathbf{I_{0+}}(e,\dot{e},n) = \int_0^{2\pi}(1 + ecos\varphi)^{n-2}[1 + (e-\dot{e})cos\varphi]^{-(n+2)}\,d\varphi =$$

$$= \int_0^{2\pi}(cos\varphi^2 + sin\varphi^2)(1 + ecos\varphi)^{n-2}[1 + (e-\dot{e})cos\varphi]^{-(n+2)}\,d\varphi =$$

$$= (e-\dot{e})^{-1}\int_0^{2\pi}cos\varphi[1 + (e-\dot{e})cos\varphi - 1](1 + ecos\varphi)^{n-2}[1 + (e-\dot{e})cos\varphi]^{-(n+2)}\,d\varphi \; -$$

$$- (e-\dot{e})^{-1}\int_0^{2\pi}sin\varphi(1 + ecos\varphi)^{n-2}[1 + (e-\dot{e})cos\varphi]^{-(n+2)}\,d[1 + (e-\dot{e})cos\varphi] =$$

$$= (e-\dot{e})^{-1}\int_0^{2\pi}cos\varphi(1 + ecos\varphi)^{n-2}[1 + (e-\dot{e})cos\varphi]^{-(n+1)}\,d\varphi \; -$$

$$= - (e-\dot{e})^{-1}\int_0^{2\pi}cos\varphi(1 + ecos\varphi)^{n-2}[1 + (e-\dot{e})cos\varphi]^{-(n+2)}\,d\varphi \; +$$

$$+ [(n+1)(e-\dot{e})]^{-1}\int_0^{2\pi}sin\varphi(1 + ecos\varphi)^{n-2}\,d\{\,[1 + (e-\dot{e})cos\varphi]^{-(n+1)}\} =$$

$$= (e-\dot{e})^{-1}\mathbf{I_1}(e,\dot{e},n) - (e-\dot{e})^{-2}\int_0^{2\pi}\{[1 + (e-\dot{e})cos\varphi] - 1\}(1 + ecos\varphi)^{n-2} \times$$

$$\times [1 + (e-\dot{e})cos\varphi]^{-(n+2)}\,d\varphi = [(n+1)(e-\dot{e})]^{-1}\{\,sin\varphi(1 + ecos\varphi)^{n-2} \times$$

$$\times [1 + (e-\dot{e})cos\varphi]^{-(n+1)}\Big|_0^{2\pi} - \int_0^{2\pi}[1 + (e-\dot{e})cos\varphi]^{-(n+1)}\,d\,[sin\varphi(1 + ecos\varphi)^{n-2}]\,\} =$$

$$= (e-\dot{e})^{-1}\mathbf{I_1}(e,\dot{e},n) - (e-\dot{e})^{-2}\int_0^{2\pi}(1 + ecos\varphi)^{n-2}[1 + (e-\dot{e})cos\varphi]^{-(n+1)}\,d\varphi \; +$$

$$+ (e-\dot{e})^{-2}\int_0^{2\pi}(1 + ecos\varphi)^{n-2}[1 + (e-\dot{e})cos\varphi]^{-(n+2)}\,d\varphi \; -$$

$$- [(n+1)(e-\dot{e})]^{-1}\int_0^{2\pi} cos\varphi(1+ecos\varphi)^{n-2}\,[1+(e-\dot{e})cos\varphi]^{-(n+1)}\,d\varphi\ +$$

$$+ (n-2)e[(n+1)(e-\dot{e})]^{-1}\int_0^{2\pi}(1-cos^2\varphi)(1+ecos\varphi)^{n-3}[1+(e-\dot{e})cos\varphi]^{-(n+1)}\,d\varphi =$$

$$= (e-\dot{e})^{-1}\mathbf{I_1}(e,\dot{e},n) - (e-\dot{e})^{-2}\mathbf{I_0}(e,\dot{e},n) + (e-\dot{e})^{-2}\mathbf{I_{0+}}(e,\dot{e},n) -$$

$$- [(n+1)(e-\dot{e})]^{-1}\mathbf{I_1}(e,\dot{e},n) + (n-2)e[(n+1)(e-\dot{e})]^{-1}\int_0^{2\pi}(1+ecos\varphi)^{n-3}\times$$

$$\times [1+(e-\dot{e})cos\varphi]^{-(n+1)}\,d\varphi\ -$$

$$- (n-2)e[(n+1)(e-\dot{e})]^{-1}\int_0^{2\pi}cos^2\varphi(1+ecos\varphi)^{n-3}\,[1+(e-\dot{e})cos\varphi]^{-(n+1)}\,d\varphi\ .$$

The later integral in the right-hand side in the above equality can easily be computed:

$$(76)\qquad \int_0^{2\pi}cos^2\varphi(1+ecos\varphi)^{n-3}\,[1+(e-\dot{e})cos\varphi]^{-(n+1)}\,d\varphi\ =$$

$$= e^{-1}\int_0^{2\pi}cos\varphi[(1+ecos\varphi)-1](1+ecos\varphi)^{n-3}\,[1+(e-\dot{e})cos\varphi]^{-(n+1)}\,d\varphi\ =$$

$$= e^{-1}\mathbf{I_1}(e,\dot{e},n) - e^{-2}\int_0^{2\pi}[(1+ecos\varphi)-1](1+ecos\varphi)^{n-3}\,[1+(e-\dot{e})cos\varphi]^{-(n+1)}\,d\varphi\ =$$

$$= e^{-1}\mathbf{I_1}(e,\dot{e},n) - e^{-2}\mathbf{I_0}(e,\dot{e},n) + e^{-2}\mathbf{I_{0-}}(e,\dot{e},n)\ ,$$

where we have taken into account the definition (1). Substituting (76) into (75), we arrive at the following linear dependence between the four integrals $\mathbf{I_{0-}}(e,\dot{e},n)$, $\mathbf{I_{0+}}(e,\dot{e},n)$, $\mathbf{I_0}(e,\dot{e},n)$ and $\mathbf{I_1}(e,\dot{e},n)$:

$$(77)\qquad 0 = [(e-\dot{e})^{-2} - 1]\mathbf{I_{0+}}(e,\dot{e},n) + (n-2)[(n+1)(e-\dot{e})]^{-1}(e-e^{-1})\mathbf{I_{0-}}(e,\dot{e},n) +$$
$$+ \{\,(n-2)[(n+1)e(e-\dot{e})]^{-1} - (e-\dot{e})^{-2}\,\}\mathbf{I_0}(e,\dot{e},n) + \{\,(e-\dot{e})^{-1} -$$
$$- [(n+1)(\,e-\dot{e})]^{-1} - (n-2)[(n+1)(\,e-\dot{e})]^{-1}\,\}\mathbf{I_1}(e,\dot{e},n)\ .$$

We choose to estimate from the above relation the integral $\mathbf{I_1}(e,\dot{e},n)$. After the multiplication of the both sides of (77) by $(e-\dot{e})$, the result is:

$$(78)\qquad 2(n+1)^{-1}\mathbf{I_1}(e,\dot{e},n) = (n-2)(1-e^2)[(n+1)e]^{-1}\mathbf{I_{0-}}(e,\dot{e},n) +$$
$$+ [(e-\dot{e})^2 - 1](\,e-\dot{e})^{-1}\mathbf{I_{0+}}(e,\dot{e},n) + \{\,(e-\dot{e})^{-1} - (n-2)[(n+1)e]^{-1}\,\}\mathbf{I_0}(e,\dot{e},n)\ .$$

Another multiplication by $(n+1)e(e-\dot{e})$ leads to the next evaluation:

$$(79)\qquad 2e(e-\dot{e})\,\mathbf{I_1}(e,\dot{e},n) = (n-2)(e-\dot{e})(\,1-e^2)\,\mathbf{I_{0-}}(e,\dot{e},n) +$$
$$+ (n+1)e[(e-\dot{e})^2 - 1]\,\mathbf{I_{0+}}(e,\dot{e},n) + [3e + (n-2)\dot{e}]\,\mathbf{I_0}(e,\dot{e},n)\ .$$

We note that (79) is derived under the assumptions $n \neq -1$, $e(u) \neq 0$ (for $|e(u)| < 1$) and $[e(u) - \dot{e}(u)] \neq 0$ (for $|e(u) - \dot{e}(u)| < 1$), which guarantees that the denominators into the expressions (75) – (78) will also be different from zero. As in the previous cases, concerning the integrals $\mathbf{I}_4(e,\dot{e},n)$ and $\mathbf{I}_2(e,\dot{e},n)$, the derived equality (79) makes sense even if these limitations are not fulfilled for a certain value $u \equiv ln\ p$. We now shall check this statement.

3.3.1. Case $n \neq -1$, $e(u) = 0$, $e(u) - \dot{e}(u) = 0$ => $e(u) = \dot{e}(u) = 0$.

The equality (79) is obviously true, because the both sides are equal to zero.

3.3.2.1. Case $n = 2$, $e(u) = 0$, $e(u) - \dot{e}(u) \neq 0$ => $\dot{e}(u) \neq 0$.

Again (79) has both sides equal to zero.

3.3.2.2. Case $n \neq -1$, $n \neq 2$, $e(u) = 0$, $e(u) - \dot{e}(u) \neq 0$ => $\dot{e}(u) \neq 0$.

The relation (79) in this case can be written as:

(80) $$0 = -(n-2)\dot{e}\mathbf{I}_{0-}(0,\dot{e},n) + (n-2)\dot{e}\mathbf{I}_0(0,\dot{e},n)\ .$$

Because $(n-2)\dot{e} \neq 0$, we may divide by this quantity, to obtain:

(81) $$\mathbf{I}_{0-}(0,\dot{e},n) - \mathbf{I}_0(0,\dot{e},n) = 0\ .$$

Using definitions (1) and (3), direct computation shows that:

(82) $$\mathbf{I}_{0-}(0,\dot{e},n) = \int_0^{2\pi} (1 - \dot{e}cos\varphi)^{-(n+1)}\, d\varphi = \mathbf{I}_0(0,\dot{e},n)\ .$$

Hence, equality (81) is trivially proved and the same is true for (79).

3.3.3. Case $n \neq -1$, $e(u) \neq 0$, $e(u) - \dot{e}(u) = 0$, => $\dot{e}(u) = e(u) \neq 0$.

The linear relation (79) can be written in the following way:

(83) $$0 = -(n+1)e\mathbf{I}_{0+}(e,\dot{e}=e,n) + (n+1)\dot{e}\mathbf{I}_0(e,\dot{e}=e,n)\ .$$

Because $\dot{e}(u) = e(u) \neq 0$ and $(n+1) \neq 0$,we can divide the both sides by $(n+1)e$:

(84) $$\mathbf{I}_0(e,\dot{e}=e,n) - \mathbf{I}_{0+}(e,\dot{e}=e,n) = 0\ .$$

From definitions (2) and (3) we directly compute that:

(85) $$\mathbf{I}_0(e,\dot{e}=e,n) = \int_0^{2\pi} (1 + ecos\varphi)^{n-2}\, d\varphi = \mathbf{I}_{0+}(e,\dot{e}=e,n)\ .$$

That is, equality (84) and, hence, the linear relation (79) are also true in that case.

3.3.4. Case $n = -1$, $e(u) = 0$, $e(u) - \dot{e}(u) = 0$ => $\dot{e}(u) = 0$.

The equality (79) now becomes 0 = 0 and it is trivially fulfilled.

3.3.5. Case $n = -1$, $e(u) = 0$, $e(u) - \dot{e}(u) \neq 0$ => $\dot{e}(u) \neq 0$.

The relation (79) now becomes:

(86) $0 = 3\dot{e}\mathbf{I}_{0\text{-}}(0,\dot{e},-1) - 3\dot{e}\mathbf{I}_0(0,\dot{e},-1)$, or $\mathbf{I}_{0\text{-}}(0,\dot{e},-1) = \mathbf{I}_0(0,\dot{e},-1)$.

We have $\mathbf{I}_0(0,\dot{e},-1) = \int_0^{2\pi} d\varphi = 2\pi$ and $\mathbf{I}_{0\text{-}}(0,\dot{e},-1) = \int_0^{2\pi} d\varphi = 2\pi$. Therefore, (86) and correspondingly (79) are satisfied.

3.3.6. Case $n = -1$, $e(u) \neq 0$, $e(u) - \dot{e}(u) = 0$ => $\dot{e}(u) = e(u) \neq 0$.

In this particular case, the both sides of the equality (79) are equal to zero.

3.3.7. Case $n = -1$, $e(u) \neq 0$, $e(u) - \dot{e}(u) \neq 0$.

The relation (79) now can be written as:

(87) $2e(e-\dot{e})\mathbf{I}_1(e,\dot{e},-1) = -3(e-\dot{e})(1-e^2)\mathbf{I}_{0\text{-}}(e,\dot{e},-1) + 3(e-\dot{e})\mathbf{I}_0(e,\dot{e},-1)$.

Because $(e - \dot{e}) \neq 0$, we may divide (87) by $(e - \dot{e})$ and check the validity of the equality:

(88) $2e\mathbf{I}_1(e,\dot{e},-1) = -3(1-e^2)\mathbf{I}_{0\text{-}}(e,\dot{e},-1) + 3\mathbf{I}_0(e,\dot{e},-1)$.

In an earlier paper [11], we have already computed for $n = -1$ in an explicit form the following analytical expressions for the integrals $\mathbf{I}_0(e,\dot{e},-1)$, $\mathbf{I}_1(e,\dot{e},-1)$ and $\mathbf{I}_{0\text{-}}(e,\dot{e},-1)$ (see in [11] formulas (2a), (2b) and (2h), respectively):

(89) $\mathbf{I}_0(e,\dot{e},-1) = \pi(2+e^2)(1-e^2)^{-5/2}$,

(90) $\mathbf{I}_1(e,\dot{e},-1) = -3\pi e(1-e^2)^{-5/2}$,

(91) $\mathbf{I}_{0\text{-}}(e,\dot{e},-1) = \pi(2+3e^2)(1-e^2)^{-7/2}$.

Consequently:

(92) $2e\mathbf{I}_1(e,\dot{e},-1) = -6\pi e^2(1-e^2)(1-e^2)^{-7/2}$,

(93) $-3(1-e^2)\mathbf{I}_{0\text{-}}(e,\dot{e},-1) + 3\mathbf{I}_0(e,\dot{e},-1) = -3\pi(2+3e^2)(1-e^2)(1-e^2)^{-7/2} + 3\pi(2+e^2)(1-e^2)(1-e^2)^{-7/2} = -6\pi e^2(1-e^2)(1-e^2)^{-7/2}$.

The right-hand sides of (92) and (93) are equal, and, hence, the linear relation (87) is proved. Thus, the reliability of the linear relation (79) is shown to remain valid in the general case of *integer*/*noninteger* powers n, $e(u)$ *equal* or *not equal* to zero (for $|e(u)| < 1$) and $[e(u) - \dot{e}(u)]$ *equal* or *not equal* to zero (for $|e(u) - \dot{e}(u)| < 1$) for *arbitrary* values of $u \equiv \ln p$. These conditions may be separately or simultaneously encountered, and the equality (79) may be used without specifying any restrictions, like the above considered.

4. Conclusions

In a series of papers ([10] – [13]), we have investigated the dynamical equation (4), governing the structure of the ***stationary elliptical*** discs in the model developed by Lyubarskij et al. [4]. Our main goal is to reveal the properties of this second order ordinary differential equation in a fully (as far as possible) analytical manner, without introducing any additional simplifications into the model, except these which already exist in the original development [4]. The first successful step in this direction was the establishment that the dynamical equation is (in the most general case) a homogeneous differential equation [12]. The next step in the simplification of the equation was to eliminate ***four*** among the ***seven*** integrals, entering as functions of $e(u)$, $\dot{e}(u)$ and n into equation (4). *In the **present** paper we pointed out how to do so with **three** of them, namely*: $\mathbf{I_4}(e,\dot{e},n)$, $\mathbf{I_2}(e,\dot{e},n)$ and $\mathbf{I_1}(e,\dot{e},n)$. *The elimination of the **fourth** integral* $\mathbf{I_0}(e,\dot{e},n)$ *will be considered in a **forthcoming** paper*. As a final result, they may be represented, by means of ***linear*** relations, through two integrals, namely: $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$. The later two integrals may be shown to be linearly independent functions on $e(u)$ and $\dot{e}(u)$ for every fixed (physically reasonable) value of the power n in the viscosity law $\eta = \beta\, \Sigma^n$. This statement *will also be proved in a **forthcoming** paper*. The problem with the integral $\mathbf{I_3}(e,\dot{e},n)$ ***still remains unresolved***. It is unclear are the three integrals (***considered together***) $\mathbf{I_{0-}}(e,\dot{e},n)$, $\mathbf{I_{0+}}(e,\dot{e},n)$ and $\mathbf{I_3}(e,\dot{e},n)$ linearly independent, or, opposite, the later integral can also be expressed as a linear combination of $\mathbf{I_{0-}}(e,\dot{e},n)$ and $\mathbf{I_{0+}}(e,\dot{e},n)$. This matter relates to the main aim of our investigations. Namely, to express the dynamical equation (4) as a sum of several terms, each factorized as a product of one of these ***two*** (or, may be, ***three***) ***linearly independent*** integrals and coefficients, which are functions on $e(u)$, $\dot{e}(u)$ and n. The linear independence would imply nullification of the coefficients. This leads to splitting of the equation (4) into a system of *probably* more simple differential equations about the ***unknown*** function $e(u)$.

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ТЪНКИ ВИСКОЗНИ ЕЛИПТИЧНИ АКРЕЦИОННИ ДИСКОВЕ С ОРБИТИ, ИМАЩИ ОБЩА ДЪЛЖИНА НА ПЕРИАСТРОНА. V. ЛИНЕЙНИ ЗАВИСИМОСТИ МЕЖДУ УСРЕДНЕНИТЕ ПО АЗИМУТАЛНИЯ ЪГЪЛ МНОЖИТЕЛИ В ДИНАМИЧНОТО УРАВНЕНИЕ

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Резюме

Ние разглеждаме модел на ***стационарни*** елиптични акреционни дискове, разработен от Любарски и др. [4], които са получили обикновено диференциално уравнение от втори ред, описващо пространствената структура на тези обекти. Това динамично уравнение съдържа седем интеграли, възникващи при азимуталното усредняване по протежение на елиптичните орбити на частиците от диска. Те са функции на ***неизвестното*** разпределение на ексцентрицитета $e(u)$, неговата производна $\dot{e}(u) \equiv de(u)/du$ и степенния показател n в закона за вискозитета $\eta = \beta \Sigma^{n}$, където $u \equiv \ln p$, p е фокалният параметър на конкретната елиптична орбита на частицата. В настоящата статия, ние извеждаме линейни зависмости между тези ***неизвестни*** интеграли, които могат да бъдат полезни за елиминирането на ***три*** от тези величини. Възможно е да бъде елимиран допълнително още един интеграл, но доказването на това твърдение ще бъде отложено в една предстояща статия. Разгледаният подход е поддържан с цел да се разцепи динамичното уравнение на една система от по-прости диференциални уравнения.