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THIN VISCOUS ELLIPTICAL ACCRETION DISCS WITH ORBITS SHARING A COMMON LONGITUDE OF PERIASTRON. IV. PROOF OF THE HOMOGENEITY OF THE DYNAMICAL EQUATION, GOVERNING DISC STRUCTURE FOR ARBITRARY POWERS *n* IN THE VISCOSITY LAW $\eta = \beta\,\Sigma^n$

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Abstract

We consider the models of elliptical accretion discs developed by Lyubarskij et al. [1] and discuss their specific properties. In particular, we emphasize on possible deviations from the Keplerian rotation, magnetorotational instability, external illumination of the disc, etc., which may take place with real accretion flows (as indicated by a lot of observations), but are not taken into account in the above theoretical constructions. According to the models, the viscosity coefficient η is adopted to have a power law form: $\eta = \beta\,\Sigma^n$ (β is a constant, Σ is the disc surface density). We investigate the dynamical equation, which is derived by Lyubarskij et al. [1], for a continuous set of values of the power n. Physically reasonable n occupy the zone between ≈ -1 and ≈ +3. The basic result of our investigation is that the dynamical equation, governing the structure of elliptical discs, is a homogeneous second order ordinary differential equation for any values of n in the designated interval. This is a generalization of our previously established result for the case of integer values of n only.

Introduction

In this paper we continue the investigation of elliptical accretion discs described by Lyubarskij et al. [1] about 15 years ago. The essential feature of their model is that not only the disc particles have *elliptical* orbits, but also the apse lines of all orbits are in line with each other. It would be stressed that it provides for changes in the eccentricity between adjacent streamlines. In other words, the eccentricities of the particle orbits may (in principle) vary for different parts of the accretion disc – its inner, middle and outer regions. The motion of the particles along the ellipses is supposed to be Keplerian and the dynamics of the disc is treated in the framework of viscous hydrodynamics. In fact, the above cited work [1] generalizes to some extent the well known α-disc model of Shakura and Sunyaev [2], where all particle orbits are *a priori* assumed to be circular. We shall consider in our investigation only the case of *stationary* accretion discs. The simplifications applied in the paper [1] lead to a second-order ordinary differential equation, which governs the structure of the accretion disc, namely, the dependence of eccentricity on the focal parameter p of the corresponding ellipse. The situation is not so favourable, in view of simplicity, in the more general models of elliptical discs considered by Ogilvie [3]. The base line followed by us is to study *analytically* these objects as much as possible in order to minimize the application of numerical methods and eventually, reveal more clearly some properties of the solutions. Despite of the property that the Ogilvie's model [3] is much more realistic and more appropriate to be checked by means of astronomical observations than earlier model of Lyubarskij et al. [1], we shall concentrate on the latter for the above mentioned reasons. Some results concerning the problem of solving of the dynamical equation for the accretion disc model of Lyubarskij et al. [1] were already published earlier in papers [4–9]. The present research may be considered as a supplement to the case of *integer* powers n [7] in the sense that now we shall consider arbitrary (but physically reasonable!) values of n, which *may not be integers*. More specifically, it will be allowed that $n \neq -1, 0, +1, +2, +3$. Of course, between integer values, n may vary continuously. We remind that, according to the adopted viscosity law $\eta = \beta \Sigma^n$ in the accretion disc model of Lyubarskij et al. [1], n is the power of the disc surface density Σ and it is allowed to take continuous (but later on fixed!) values in some appropriate interval. The more general situation, when the power n eventually depends on some physical and geometrical factors like pressure, temperature, volume density, distance to the compact object, shape of the accretion disc and so on, is out of the scope of our investigation. Henceforth, the quantity n (and also the factor β) are considered to be constants and the variation of the viscosity coefficient η throughout the disc is determined by the variation of its surface density Σ.

With the accumulation of observational data about accretion discs in binary stellar systems, it becomes more and more evident that the ellipticity of these objects is not only a possible theoretical construction, but it is often a phenomenon occurring in reality. The overview paper [10] gives some examples on this subject. We shall now add to this list of papers several more recent results concerning the theme of elliptical discs. Especially, we shall mention the photometric observations of Howell et al. [11] in the infrared region of the electromagnetic spectrum of the eclipsing interacting binary WZ Sge. There is evidence that the accretion flow in this stellar system is much more complex than previously suggested. The astronomical observations support the conclusion that that the so-far-known *gaseous* accretion disc is surrounded by an *asymmetric* disc of *dusty* material with a radius about 15 times larger than the outer radius of the gaseous disc. The dust disc contains only a small amount of mass and is completely invisible at optical and near-infrared wavelengths. Such discovery in respect to the structure of the *global* (gas + dust) disc has significant influence on accretion disc theory. It suggests the need of certain generalization of the model, advocated by Lyubarskij et al. [1], to “two-component disc” case, but we shall not deal with this problem here. Indications of the presence of an elliptical accretion disc are found in the X-ray binary star UW Coronae Borealis [12]. The interpretation of the light curve for this system is based on the assumption of eclipse of an accretion disc around a neutron star by the secondary star. The surface brightness of the accretion disc is strongly *asymmetric* and highly variable. Observations show that the variations of eclipse morphology are repeated at a period of 5.5 days and the shape of the superhump-like modulation also varies at this period. The model developed by Mason et al. [12] assumes elliptical distribution of surface brightness and, respectively, the disc precesses at the 5.5-day period in order to reproduce the eclipse depths and the times of mideclipse. The superhump-like variability phenomenon for the UW Coronae Borealis star may be explained reasonably well by an elliptical precessing disc. The paper of Ferreira and Ogilvie [13] considers warping and eccentric disturbances propagating inwards in discs around black holes under a wide variety of conditions. It is assumed that the deformations are stationary and propagate in a disc model similarly to the regions (a) (dominant radiation pressure; electron scattering on free electrons plays the main role) and (b) (dominant gas pressure; electron scattering accounts for the main contribution to opacity) of Shakura and Sunyaev discs [2]. This investigation shows that the propagation of warping and eccentric deformations to the innermost regions of the disc is favoured by the low viscous damping and the high accretion rate.

Deviations from the Keplerian Rotation of Accretion Flows

The elliptical accretion disc model of Lyubarskij et al. [1] assumes the idea (and this concept is supported by the observations) that the discs are Keplerian, i.e., rotationally supported gaseous/dust discs. The common scenario, which naturally leads to the formation of Keplerian discs, is the viscous decretion model. According to it, discs are hydrodynamically supported in the vertical direction, while the radial structure is governed by viscous transport. This circumstance determines that the temperature is a primary factor that governs disc density structure. Of course, many physical and geometrical properties of accretion flows depend on the central stars around which discs rotate. For example, Carciofi et al. [14] discuss the basic hydrodynamics that determines the density structure of the discs around hot stars and solve the full problem of the *steady* state nonisothermal viscous diffusion and vertical hydrostatic equilibrium. They find that for Keplerian viscous discs, the self-consistent solutions depart significantly from the analytic isothermal density, which may have potentially large effects on the emergent spectrum. The implication is that for detailed modelling of Be star discs, nonisothermal disc models must be used. But in the opposite case, Shu et al. [15] find that strong magnetization makes the discs surrounding young stellar objects to rotate at rates that are too sub-Keplerian to enable the thermal launching of disc winds from their surfaces. An exception is observed when the rate of gas diffusion across field lines is dynamically fast. Another study [16] also supports the possibility of non-Keplerian motion in the inner regions of accretion discs around compact objects (i.e., the orbital frequency of the gas deviates from the *local* Keplerian value). It is shown that for long-wavelength modes in this region, the *radial* epicyclic frequency k is higher than the *azimuthal* frequency Ω. This circumstance may have significant implications for models of the twin kilohertz quasi-periodic oscillations observed in many neutron star sources. Hence, it can be subject to observational verifications. Deviations from the Keplerian velocity law are able to cause various kinds of instabilities, which are not taken into account in the model of Lyubarskij et al. [1] considered by us.

For example, in the paper of Barranko [17] it is shown that, as the dust in a protoplanetary disc settles, a vertical shear develops. The reason for this phenomenon is that the dust-rich gas in the midplane of the accretion disc orbits at a rate closer to the Keplerian velocity than the slower moving dust-depleted gas above and below the midplane. The classical analysis (i.e., neglecting the Coriolis force and differential rotation) performed by Barranko [17] predicts that Kelvin-Helmholtz instability occurs when the Richardson number of the stratified shear flow is $\leq 1/4$. Planets embedded in optically thick passive accretion discs are also expected to produce perturbations in the density and temperature structure of the disc. The calculated magnitudes of these perturbations for a range of planet masses and different radial distances from the center of the disc are given in [18]. It is demonstrated that a self-consistent calculation of the density and temperature structure of the accretion flow has great effect on the disc model. In addition, the temperature structure in the disc is highly sensitive to the angle of incidence of stellar irradiation at the surface. Therefore, the accurate calculation of the *shape* of the disc surface is crucial for modelling the disc's thermal structure. In this connection it is worthy to note that the model of Lyubarskij et al. [1] does not take into account the external illumination of elliptical accretion discs.

The investigation of the magnetorotational instability and its evolution in protoplanetary discs that have radial non-uniform magnetic field which allow stable and unstable regions to coexist *initially* is considered in paper [19]. It is found that a zone in which the disc gas rotates with a super-Keplerian velocity emerges as a result of the non-uniformly growing magnetorotational turbulence. It is also established that the original Keplerian shear flow is transformed to a quasi-steady flow such that more flattened (even with *rigid* rotation in the *extreme* cases) velocity profile emerges *locally* and the *outer* part of the velocity profile tends to be super-Keplerian. According to Kato et al. [19], angular momentum and mass transfer due to temporally generated magnetorotational instability turbulence in the initially unstable region are responsible for this transformation. Since the paper of Lyubarskij et al. [1] (and, consequently, our investigations, which are based on it) does not include the latter factor, we shall very briefly discuss in the next section how this simplification affects the accretion disc structure and how it restricts the applicability of this model. Of course, we must have in mind that the other approximations are also essential.

Significance of Magnetic Instabilities for the Realistic Description of Accretion Disc Models around Stellar Mass Objects

Let us take into consideration the work of Nekrasov [20] where electromagnetic streaming instabilities of multicomponent collisional *magnetized* accretion discs are studied. In this paper, sufficiently ionized regions of the disc are explored because there is strong collisional coupling of neutral atoms with both ions and dust grains simultaneously. The *steady state* is investigated in details and the azimuthal and radial background velocities of the species are calculated. The azimuthal velocities of ions, dust grains and neutral particles are found to be less than the Keplerian velocity. Concerning the radial velocities of the latter and the dust grains, it is shown that they are directed inward of the disc. Also, the dispersion relation for the vertical perturbations is derived and its unstable solutions, due to different background velocities of ions and electrons, are established [20]. It is found that the growth rates of the considered streaming instabilities can be much larger than the Keplerian frequency.

The accretion disc model of Lyubarskij et al. [1] is developed on the basis of classical Newtonian mechanics, i.e. general relativistic effects are not taken into account. But these may be essential in the near vicinities of compact objects. Consequently, by the notation "inner boundary"/ "innermost radius" of the accretion disc (respectively, "the smallest parameter" p_{min} (or p_{in}) of the elliptical disc) we shall mean such a value of the parameter p which marks the boundary closest to the central star, inside which the considered disc models are not already so good approximations as for the outer parts of the disc. Unfortunately, the general relativistic constraints are not the only factors that are able to impose restrictions on the reasonable validity of the model in the innermost regions of accretion flow. It is not excluded at all that other physical conditions and geometrical arguments may "augment" the value of the "inner disc radius" beyond the limit placed by the general relativistic considerations. In this sense, it is worthy to note the recent investigation of Beckwith et al. [21], which examines general relativistic magnetohydrodynamic simulations of black hole accretion. They find significant magnetic stresses near and inside the innermost stable circular orbit, which implies that such flows could radiate in a manner noticeably different from the prediction of the standard α-model of Shakura and Sunyaev [2]. The latter model assumes that in these regions there are not stresses. Beckwith et al. [21] obtain estimates of how phenomenologically important parameters like the "radiation edge" (i.e., the innermost ring of the disc from which substantial thermal radiation escapes to infinity) may be altered near the innermost stable circular orbit. Their estimates are based on a large number of 3D general relativistic magnetohydrodynamic simulations combined with general relativity ray tracing. For slowly spinning black holes, the radiation edge lies well inside where the standard α-disc model [2] predicts, particularly when the stellar system is viewed at high inclination. For more rapidly spinning black holes, the contrast is established to be smaller. It is estimated in [21] that for a fixed total luminosity, the characteristic temperature of accretion flow increases by a factor between 1.2 and 2.4 over that predicted by the standard model [2]. If the mass accretion rate is fixed, there is a corresponding enhancement of accretion luminosity, which may be anywhere from tens of percent to the order of unity [21].

In our investigation, we are dealing with *stationary* (i.e., *steady state*) models of accretion discs. But we must keep in mind that the more realistic descriptions take into account different kinds of instabilities. For this reason, we shall add some remarks concerning this item in order to emphasize to some extent the limitations inherent to our model. For example, when the accretion rate is more than a small fraction of the Eddington rate dM_{Edd}/dt, the inner regions of the accretion discs around black holes are expected to be radiation-dominated. However, in the α-models [1,2] these regions are, in addition, thermally unstable. In the 3D radiation magnetohydrodynamic simulations of a vertically stratified shearing box (ratio of radiation to gas pressure $p_{rad}/p_g \sim 10$) performed by Hirose et al. [22], no thermal runaway occurs over a timespan of 40 cooling times t_{cool}. They observe that stress and total pressure are well correlated as predicted by the α-model [2], but stress *fluctuations* precede pressure *fluctuations*, contrary to the common suggestion that pressure controls the saturation level of magnetic energy. According to [22], this circumstance determines the thermal stability of the accretion flow. When the *thin* accretion discs around black holes are perturbed, the main restoring force is gravity. The authors of paper [23] state that, if gas pressure, magnetic stresses and radiation pressure are neglected, the disc remains *thin* as long as the particle orbits do not intersect (this condition *is fulfilled by hypothesis* in the model of Lyubarskij et al. [1] !). They also find that a discrete set of perturbations is possible for which orbits remain non-intersecting for arbitrarily long times. Correspondingly, these models define a discrete set of perturbations.

According to [24], accretion discs in which angular momentum transport is dominated by magnetorotational instability may also possess additional (*purely hydrodynamic*) turbulence drivers. Even when the hydrodynamic processes themselves generate negligible levels of transport, they may still affect the disc's evolution through their influence on magnetorotational instability. The conclusion is that the impact of hydrodynamic turbulence is generically subject to ignorance only in some cases. Such a phenomenon is convection, where additional turbulence arises due to the accretion energy released by magnetorotational instability alone. Hydrodynamic turbulence may affect (either enhance or suppress) magnetorotational instability, if it is both strong and driven by independent mechanisms, such as self-gravity, supernovae explosions or solid(dust) particles-gas interactions in multiphase protoplanetary discs [24].

With the above notes about the adopted simplifications, which are intrinsic to standard α-disc models [1,2], we now begin to consider a very concrete problem, namely, the transformation of the dynamical equation for elliptical accretion discs (derived by Lyubarskij et al. [1]) to a more simple form. The main purpose of our approach is to do this in *purely analytical way*. It may come out that the final results of our attempts do not provide successfully to express this equation in a form, which allows solving it by means of the known standard methods from the theory of ordinary differential equations. But even in this pessimistic scenario, we hope that this will reveal some properties of the physical and mathematical structure of the disc model [1].

The Dynamical Equation and the Specific Characteristics of Its Terms

For clarity of subsequent computations, we rewrite here the dynamical equation governing the structure of elliptical accretion discs (with orbits of their species sharing a common longitude of periastron; [1]) in the form which was already derived in the previous paper [6]:

$$(1) \qquad \sum_{i,k} \mathbf{A}_{ik}(e,\dot{e},n)\, \mathbf{I}_i(e,\dot{e},n)\, \mathbf{I}_k(e,\dot{e},n)\, \ddot{e} + \sum_{l,m} \mathbf{B}_{lm}(e,\dot{e},n)\, \mathbf{I}_l(e,\dot{e},n)\, \mathbf{I}_m(e,\dot{e},n)\, \dot{e} = 0,$$

where the indices i, k, l and m independently take the values of 0–, 0+, 0, 1, 2, 3 and 4. The eccentricity e of the ellipse of each particle trajectory and its derivative $\dot{e} = de/du \equiv de/d(lnp)$ with respect to the logarithmic scale $u \equiv lnp$ of the focal parameter p of the ellipse, are (strictly speaking) functions of u and the power n in the viscosity law $\eta = \beta\, \Sigma^{n}$. According to the agreement made in the beginning of the present paper, we shall assume in what follows, that n is an arbitrarily chosen *fixed* non-integer number. That is to say, the quantity n does not vary along the values of the focal parameter p (the "radius" of the disc). Naturally, the convention that n is the same for all parts of the accretion disc, simplifies greatly the mathematical treatment of the problem. Consequently, for every previously selected (and after that fixed !) value of n between ≈ -1 and ≈ +3, we shall consider e and $\dot{e}$ as functions depending only on $u \equiv lnp$. We rewrite the definitions of the integrals $\mathbf{I}_{0-}$, $\mathbf{I}_{0+}$, $\mathbf{I}_0$, $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_4$ [6]:

$$(2)\qquad \mathbf{I}_{0-}(e,\dot{e},n) \equiv \int_0^{2\pi} (1 + ecos\varphi)^{n-3}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}\, d\varphi\ ,$$

$$(3)\qquad \mathbf{I}_{0+}(e,\dot{e},n) \equiv \int_0^{2\pi} (1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-(n+2)}\, d\varphi\ ,$$

$$(4)\qquad \mathbf{I}_{j}(e,\dot{e},n) \equiv \int_0^{2\pi} (cos\varphi)^{j}\, (1 + ecos\varphi)^{n-2}[1 + (e - \dot{e})cos\varphi]^{-(n+1)}\, d\varphi\ ;$$

j = 0, 1, 2, 3, 4.

The appearance of the above seven integrals in dynamical equation (1) is related to the averaging over the azimuthal angle φ, as proceeded in the model of elliptical accretion discs [1]. The other quantities to appear in equation (1) are the coefficients $\mathbf{A}_{ik}(e,\dot{e},n)$ and $\mathbf{B}_{lm}(e,\dot{e},n)$, which are known functions of e, $\dot{e}$ and n. Their writing in explicit form is too long to be given here and we shall miss this procedure in order to save unnecessary tedious formulae. We stress that the main purpose of the present paper is not to investigate the known functions $\mathbf{A}_{ik}$ and $\mathbf{B}_{lm}$, but to overcome the difficulties generated by the presence of these seven integrals, $\mathbf{I}_{0-}$, $\mathbf{I}_{0+}$, $\mathbf{I}_0$,…, $\mathbf{I}_4$, in dynamical equation (1). It must be pointed out that both the coefficients $\mathbf{A}_{ik}$ and $\mathbf{B}_{lm}$ and the integrals $\mathbf{I}_{0-}$, $\mathbf{I}_{0+}$, $\mathbf{I}_0$,…, $\mathbf{I}_4$ are, as a final result, functions of the parameter $u \equiv lnp$. But actually, at this stage of the task's solving, we do not know the solution(s) of equation (1). ***This is just our final goal*** **!** For this reason, we shall assume in what follows {e, $\dot{e}$ and n = preliminary fixed constant} to be independent variables, having however in mind that differentiation of e with respect to u (we note this by a dot ($\dot{}$)) gives $\dot{e}$. Because $\dot{e}$ may also depend on u, it is again true that $d\dot{e}/du = \ddot{e}(u)$. These properties must be taken into account when the above-considered coefficients and integrals, participating in the subject-to-simplifications equation (1) undergo differentiation with respect to $e(u)$ or $\dot{e}(u)$.

Proof that the Dynamical Equation Is a Second-Order *Homogeneous* Ordinary Differential Equation

In the original deriving of the dynamical equation, which determines the structure of elliptical discs with orbits sharing a common longitude of periastron, Lyubarskij et al. include a free term ([1], equation (45)):

(5) $\mathbf{F}(e,\dot{e},n) \equiv \mathbf{Y}[(3/2)\mathbf{W} - \mathbf{Z} - (1/2)(1 - e^2)\mathbf{Y}]$,

The bars above **Y**, **Z** and **W** (which are present in paper [1], but are *omitted* in our notations for typographical technical reasons) denote averaging over the azimuthal angle φ. At first sight, it seems that this expression (5) stipulates the dynamical equation ([1], equation (45)) to be an inhomogeneous differential equation. For accretion discs with constant eccentricity of particle orbits ($\dot{e}(u) \equiv 0$; correspondingly $\ddot{e}(u) \equiv 0$ throughout the disc) it is obvious from ([1],equation (45)) that the free term $\mathbf{F}(e,\dot{e},n)$ (5) vanishes in a similar way for the entire disc. In the earlier paper [7], the explicit expressions for the auxiliary functions $\mathbf{Y}(e,\dot{e},n)$, $\mathbf{Z}(e,\dot{e},n)$ and $\mathbf{W}(e,\dot{e},n)$ were given for *integer* values of n (n = -1, 0, +1, +2, +3) ([7], equations (7a) – (11c)). Proving the homogeneity of the dynamical equation in the latter case was based on a little different approach. There is no necessity for the free term $\mathbf{F}(e,\dot{e},n)$ to be identically equal to zero. It is enough to show that $\mathbf{F}(e,\dot{e},n)$ factorizes and one of the factors is just the derivative of the eccentricity $\dot{e}(u) \equiv de/du$. Further on, $\mathbf{F}(e,\dot{e},n)$ can be "absorbed" into the term $\{\mathbf{Y}(e,\dot{e},n)[\partial\mathbf{Z}(e,\dot{e},n)/\partial e] - \mathbf{Z}(e,\dot{e},n)[\partial\mathbf{Y}(e,\dot{e},n)/\partial e] - \mathbf{Y}^2(e,\dot{e},n)e\}\dot{e}$ ([1], equation (45)), which also contains as a common factor the first derivative $\dot{e}(u)$. This implies that equation (45), established by Lyubarskij et al. [1], is in fact, a second-order *homogeneous* ordinary differential equation. In paper [7], factorization of the free term $\mathbf{F}(e,\dot{e},n)$ is evident from its explicit computation thanks to the already analytically evaluated expressions for integrals $\mathbf{I}_{0-}$, $\mathbf{I}_{0+}$, $\mathbf{I}_0$, $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_4$ (see equations

(2a)–(6h) from [7]). There, it was only noticed that the proof of the homogeneity of the dynamical equation may be also generalized to the case of non-integer values of the power *n*. We shall now give a proof of this statement. We shall take into account the circumstance that we do not know (at least at the present time!) any analytical evaluations for non-integer *n* of the integrals $\mathbf{I}_{0-}$, $\mathbf{I}_{0+}$, $\mathbf{I}_0$, $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_4$, defined by relations (2), (3) and (4).

Let us start with detailed derivation of the expressions for $\mathbf{Y}(e,\dot{e},n)$, $\mathbf{Z}(e,\dot{e},n)$ and $\mathbf{W}(e,\dot{e},n)$. *Here we make an important note***.** For technical/ typographical reasons, we denote by $\mathbf{Y}(e,\dot{e},n)$, $\mathbf{Z}(e,\dot{e},n)$ and $\mathbf{W}(e,\dot{e},n)$ the corresponding angle-averaged functions defined by equation (42) in the paper of Lyubarskij et al. [1], and denoted by barred gothic capital letters. This is, of course, evident when we write the free term $\mathbf{F}(e,\dot{e},n)$ (5) in our paper and compare it with the notations in the dynamical equation (45) in [1]. Consequently, our notations $\mathbf{Y}(e,\dot{e},n)$, $\mathbf{Z}(e,\dot{e},n)$ and $\mathbf{W}(e,\dot{e},n)$ *must not be erroneously identified* with the same designated quantities in [1] (equations (35), (36) and (38)). The latest we shall rewrite as:

(6) $\mathbf{Y}_1(e, \dot{e}, \varphi) \equiv -(2/3)(GMp)^{-1/2}\, g\, r^p \sigma^{p\varphi}$,

(7) $\mathbf{Z}_1(e, \dot{e}, \varphi) \equiv -(2/3)(GM/p)^{-1} \mathrm{V}_{\mathrm{i}} \sigma^{p\mathrm{i}}\, g^{1/2}$,

(8) $\mathbf{W}_1(e, \dot{e}, \varphi) \equiv (2/9)(GM/p^2)^{-1} \sigma_{\mathrm{i\,k}}\, \sigma^{\mathrm{i\,k}}\, g^{1/2}$.

Note that these functions $\mathbf{Y}_1$, $\mathbf{Z}_1$ and $\mathbf{W}_1$ *do not depend* on the power *n* (see expressions (A5), (A7), (A12), (A14) and (A16) from Appendix A of [1]; see also formulae (9), (10) and (11) bellow in this paper). In the above formulae (6)–(8), *G* is the Newton's gravitational constant, *M* is the mass of the central star around which the accretion disc rotates, *p* is the focal parameter of the considered elliptical particle orbit. To describe the problem, non-orthogonal curvilinear coordinates (*p*, *φ*) are used instead of Cartesian ones (x, y). Correspondingly, V_{i} (i = *p*, *φ*) are the covariant components of the Keplerian velocity, $\sigma_{\mathrm{i\,k}}$ and $\sigma^{\mathrm{i\,k}}$ (i, k = *p*, *φ*) are the covariant and contravariant components of the shear tensor $\boldsymbol{\sigma}$, *g* is the determinant of the metric tensor and r^p is the contravariant *p*-component of the radius vector $\boldsymbol{r}$. We perform in advance: (i) correction of some typographical errors and (ii) simplification of some expressions in Appendix A to Lyubarskij et al. [1]. For example, the right-hand side of the formula for the trace of the shear tensor $\sigma \equiv \sigma_{\mathrm{i\,k}}\, \sigma^{\mathrm{i\,k}}$ ([1], equation (A16)) may be factorized. The substitution of the final results in (6)–(8) gives the following representations about the auxiliary functions:

(9) $3\mathbf{Y}_1(e, \dot{e}, \varphi) = (1 + e cos\varphi)^{-2}[1 + (e - \dot{e})cos\varphi]^{-1}(3 + 7e cos\varphi + e^2 + 4e^2cos^2\varphi + e^3cos\varphi - 4\dot{e}cos\varphi + 2e\dot{e} - 8e\dot{e}cos^2\varphi - 2e^2\dot{e}cos\varphi)$,

(10) $3\ \mathbf{Z}_1(e, \dot{e}, \varphi) = (1 + e cos\varphi)^{-2}[1 + (e - \dot{e})cos\varphi]^{-1}(3 + 13e cos\varphi + 22e^2cos^2\varphi + 2e^3cos\varphi + 16e^3cos^3\varphi + e^4 + 2e^4cos^2\varphi + 4e^4cos^4\varphi + e^5cos\varphi - 4\dot{e}cos\varphi - 2e\dot{e} - 12e\dot{e}cos^2\varphi - 6e^2\dot{e}cos\varphi - 12e^2\dot{e}cos^3\varphi - 2e^3\dot{e} - 4e^3\dot{e}cos^2\varphi - 2e^4\dot{e}cos\varphi - 4e^3\dot{e}cos^4\varphi)$,

(11) $9\ \mathbf{W}_1(e, \dot{e}, \varphi) = (1 + e cos\varphi)^{-2}[1 + (e - \dot{e})cos\varphi]^{-1}(9 + 33e cos\varphi - 2e^2cos^2\varphi - 2\ e^3cos\varphi + 32e^3cos^3\varphi + e^4 + 8e^4cos^4\varphi + e^5cos\varphi - 24\dot{e}cos\varphi + 4e\dot{e} - 72e\dot{e}cos^2\varphi + 4e^2\dot{e}cos\varphi - 72e^2\dot{e}cos^3\varphi - 4e^3\dot{e} - 24e^3\dot{e}cos^4\varphi - 4e^4\dot{e}cos\varphi + 8\dot{e}^2 + 8\dot{e}^2cos^2\varphi + 8e\dot{e}^2cos\varphi + 24e\dot{e}^2cos^3\varphi + 4e^2\dot{e}^2 + 16e^2\dot{e}^2cos^4\varphi + 4e^3\dot{e}^2cos\varphi)$.

From Appendix A ([1], equations (A5) and (A12)), we also compute:

(12) $(g^{1/2}\ V^{\varphi})^n = (GM/p)^{n/2}\ (1 + e cos\varphi)^{-n}[1 + (e - \dot{e})cos\varphi]^n$.

Then, following definitions (40) and (42) from the paper of Lyubarskij et al. [1], we can write the new auxiliary functions (depending already on n):

(13) $\mathbf{Y}_2(e, \dot{e}, n, \varphi) \equiv \mathbf{Y}_1(e, \dot{e}, \varphi)(g^{1/2}\ V^{\varphi})^{-n}$,

(14) $\mathbf{Z}_2(e, \dot{e}, n, \varphi) \equiv \mathbf{Z}_1(e, \dot{e}, \varphi)(g^{1/2}\ V^{\varphi})^{-n}$,

(15) $\mathbf{W}_2(e, \dot{e}, n, \varphi) \equiv \mathbf{W}_1(e, \dot{e}, \varphi)(g^{1/2}\ V^{\varphi})^{-n}$,

We stress again that in our notations the symbol "dot" ($\dot{}$) means differentiation with respect to the variable $u \equiv lnp$, ***not*** with respect to the time t. Finally, the averaging over the azimuthal angle φ yields the expressions for $\mathbf{Y}(e,\dot{e},n)$, $\mathbf{Z}(e,\dot{e},n)$ and $\mathbf{W}(e,\dot{e},n)$ we are seeking for:

(16) $3\mathbf{Y}(e,\dot{e},n) \equiv (3/(2\pi))\int_0^{2\pi}\mathbf{Y}_2(e, \dot{e}, n, \varphi)d\varphi = (2\pi)^{-1}(p/GM)^{n/2}[(3 + e^2 + 2e\dot{e})\mathbf{I}_0 + (7e + e^3 - 4\dot{e} - 2e^2\dot{e})\mathbf{I}_1 + (4e^2 - 8e\dot{e})\mathbf{I}_2]$,

(17) $3\mathbf{Z}(e,\dot{e},n) \equiv (3/(2\pi))\int_0^{2\pi}\mathbf{Z}_2(e, \dot{e}, n, \varphi)d\varphi = (2\pi)^{-1}(p/GM)^{n/2}[(3 + e^4 - 2e\dot{e}$

$-2e^3\dot{e})\mathbf{I}_0 + (13e + 2e^3 + e^5 - 4\dot{e} - 6e^2\dot{e} - 2e^4\dot{e})\mathbf{I}_1 + (22e^2 + 2e^4 - 12e\dot{e} - 4e^3\dot{e})\mathbf{I}_2 + (16e^3 - 12e^2\dot{e})\mathbf{I}_3 + (4e^4 - 4e^3\dot{e})\mathbf{I}_4]$,

(18) $$9\mathbf{W}(e,\dot{e},n) \equiv (9/(2\pi))\int_0^{2\pi}\mathbf{W}_2(e, \dot{e}, n, \varphi)d\varphi = (2\pi)^{-1}(p/GM)^{n/2}[(9 - 2e^2 + e^4 + 4e\dot{e} - 4e^3\dot{e} + 8\dot{e}^2 + 4e^2\dot{e}^2)\mathbf{I}_0 + (33e - 2e^3 + e^5 - 24\dot{e} + 4e^2\dot{e} - 4e^4\dot{e} + 8e\dot{e}^2 + 4e^3\dot{e}^2)\mathbf{I}_1 + (48e^2 - 72e\dot{e} + 8\dot{e}^2)\mathbf{I}_2 + (32e^3 - 72e^2\dot{e} + 24e\dot{e}^2)\mathbf{I}_3 + (8e^4 - 24e^3\dot{e} + 16e^2\dot{e}^2)\mathbf{I}_4] .$$

In deriving the last three expressions, we took into account definitions (4) of the integrals $\mathbf{I}_j(e, \dot{e}, n)$, (j = 0, 1, 2, 3, 4). Having already the results (16)–(18), it is simple to calculate the free term $\mathbf{F}(e,\dot{e},n)$:

(19) $$18\mathbf{F}(e,\dot{e},n) \equiv (3\mathbf{Y})[9\mathbf{W} - 2(3\mathbf{Z}) + (e^2 - 1)(3\mathbf{Y})] = (2\pi^2)^{-1}(p/GM)^{n}\,\boldsymbol{\dot{e}}\,[(9e + 6e^3 + e^5 + 12\dot{e} + 16e^2\dot{e} + 4e^4\dot{e} + 8e\dot{e}^2 + 4e^3\dot{e}^2)\mathbf{I}_0^2 + (-18 + 36e^2 + 14e^4 + 16e\dot{e} + 32e^3\dot{e} - 16\dot{e}^2 - 8e^2\dot{e}^2)\mathbf{I}_1\mathbf{I}_0 + (-42e + 43e^3 - e^7 + 24\dot{e} + 12e^2\dot{e} + 8e^4\dot{e} + 4e^6\dot{e} - 16e\dot{e}^2 - 16e^3\dot{e}^2 - 4e^5\dot{e}^2)\mathbf{I}_1^2 + (-60e - 8e^3 + 4e^5 + 12\dot{e} - 44e^2\dot{e} - 24e\dot{e}^2 - 16e^3\dot{e}^2)\mathbf{I}_2\mathbf{I}_0 + (-164e^2 + 8e^4 - 4e^6 + 156e\dot{e} + 4e^3\dot{e} + 16e^5\dot{e} - 16\dot{e}^2 - 40e^2\dot{e}^2 - 16e^4\dot{e}^2)\mathbf{I}_2\mathbf{I}_1 + (-80e^3 + 176e^2\dot{e} - 32e\dot{e}^2)\mathbf{I}_2^2 + (-72e^2 - 24e^4 + 36e\dot{e} - 36e^3\dot{e} + 24e^2\dot{e}^2)\mathbf{I}_3\mathbf{I}_0 + (-168e^3 - 24e^5 + 180e^2\dot{e} + 60e^4\dot{e} - 48e\dot{e}^2 - 24e^3\dot{e}^2)\mathbf{I}_3\mathbf{I}_1 + (-96e^4 + 240e^3\dot{e} - 96e^2\dot{e}^2)\mathbf{I}_3\mathbf{I}_2 + (-24e^3 - 8e^5 + 24e^2\dot{e} - 8e^4\dot{e} + 16e^3\dot{e}^2)\mathbf{I}_4\mathbf{I}_0 + (-56e^4 - 8e^6 + 88e^3\dot{e} + 24e^5\dot{e} - 32e^2\dot{e}^2 - 16e^4\dot{e}^2)\mathbf{I}_4\mathbf{I}_1 + (-32e^5 + 96e^4\dot{e} - 64e^3\dot{e}^2)\mathbf{I}_4\mathbf{I}_2] .$$

The above complex and lengthy expression shows clearly that a common factor $\boldsymbol{\dot{e}}$ appears when factorization has been performed. It is worthy to note that we have not imposed any restrictions on the power n in the viscosity law $\eta = \beta\,\Sigma^{n}$. That is to say, the purely analytically derived result (19) is valid both for integer and non-integer values of n. As a final result, the "free" term $\mathbf{F}(e,\dot{e},n)$, given by definition (5), can be absorbed into the term $(\mathbf{Y}\partial\mathbf{Z}/\partial e - \mathbf{Z}\partial\mathbf{Y}/\partial e - \mathbf{Y}^2 e)\boldsymbol{\dot{e}}$ of the dynamical equation ([1], equation (45); call to mind that we are using other notations in the present paper, as mentioned earlier). This completes the proof that the dynamical equation, which determines the structural properties of stationary elliptical accretion discs *with apse lines of all orbits assumed to be in line with each other*, is a second-order *homogeneous* ordinary differential equation. This argument was taken into account when we wrote the dynamical equation in form (1).

Conclusions

As illustrated by relation (19), the coefficients $\mathbf{A}_{ik}(e, \dot{e}, n)$ and $\mathbf{B}_{lm}(e, \dot{e}, n)$, (i, k, l, m = 0–, 0+, 0, 1, 2, 3, 4) are expected to be very complex functions of their arguments e, $\dot{e}$ and n. Consequently, the homogeneity of dynamical equation (1) is far from the condition to be a sufficiently simplifying property, allowing for its easy (or without great difficulties) solving by means of purely analytical methods. For this reason, the necessity arises to find additional simplifications of equation (1), though there are no optimistic indications for such useful possibilities. Even in the case of integer values of the power n in the viscosity law $\eta = \beta\, \Sigma^{n}$, when each of the seven integrals $\mathbf{I}_{0-}$, $\mathbf{I}_{0+}$, $\mathbf{I}_0$, **…,** $\mathbf{I}_4$ may be found in an explicit analytical form, the dynamical equation can hardly be approximated by a differential equation with constant coefficients [7]. One possible approach to simplify further this equation is to try to find relations between these seven integrals, allowing us to exclude some of them in equation (1). In the simplest case we are to seek for linear ones. The attractivity of this idea increases when non-integer powers n are considered, because (at least at the present time) we do not even know any explicit analytical solutions of the integrals $\mathbf{I}_{0-}$, $\mathbf{I}_{0+}$, $\mathbf{I}_0$, **…,** $\mathbf{I}_4$. Dynamical equation (1) determines the structure of elliptical discs with particles sharing a common longitude of periastron. If the attempts to solve it to the very end by purely analytical methods appear to be unsuccessful, we nevertheless hope that the attained simplifications will allow us to accomplish some more problems. For example, the number of branch points of the solutions, the limitations imposed on the domains of the solutions, certain qualitative characteristics of theirs, etc. Of course, analytically simplified forms of (1) are probably preferred when numerical methods for solving are used.

In conclusion, we would line to make the following remark. Above we have written definitions (2) and (3) of the integrals $\mathbf{I}_{0-}(e, \dot{e}, n)$ and $\mathbf{I}_{0+}(e, \dot{e}, n)$, accordingly. At first sight, it may seem that this is an unnecessary supplement to definitions (4) of the integrals $\mathbf{I}_j(e, \dot{e}, n)$, (j = 0,1,…, 4). In fact, this information was not used when deriving the final expression (19). This is so because the latter result serves only to reveal the property that a factor $\dot{e}$ appears, when the free term $\mathbf{F}(e, \dot{e}, n)$ is factorized. For this purpose, we do not perform any differentiations of integrals $\mathbf{I}_j(e, \dot{e}, n)$, (j = 0,1,…, 4). Such operations will lead to the appearance of $\mathbf{I}_{0-}(e, \dot{e}, n)$ and $\mathbf{I}_{0+}(e, \dot{e}, n)$, and, consequently, of their presence in dynamical equation (1). It can also be seen in explicit form, if we try to eliminate some of the integrals $\mathbf{I}_0$, $\mathbf{I}_1$,

…, $\mathbf{I}_4$ by deriving linear relations between them, including differentiation operations. This will be subject of a forthcoming paper and we provide definitions (2) and (3) mostly for completeness of the discussion.

References

1. L y u b a r s k i j Y u. E., K. A. P o s t n o v, M. E. P r o k h o r o v. Eccentric accretion discs., Monthly Not. Royal Astron. Soc., 266, 1994, № 2, p. 583–596.
2. S h a k u r a N. I., R. A. S u n y a e v. Black holes in binary systems. Observational appearance., Astron. & Astrophys., 24, 1973, № 3, p. 337–355.
3. O g i l v i e G. I. Non–linear Dynamics of eccentric discs., Monthly Not. Royal Astron. Soc., 325, 2001, № 1, p. 231–248.
4. D i m i t r o v D. V. Elliptical accretion discs with constant eccentricity. I. Case $\eta = \beta\,\Sigma^n$., Aerospace Research in Bulgaria, 14, 1998, p. 3–10.
5. D i m i t r o v D. V. Elliptical accretion discs with constant eccentricity. II. Standard α-disc model., Aerospace Research in Bulgaria, 15, 1999, p. 11–21.
6. D i m i t r o v D. V. One possible simplification of the dynamical equation governing the evolution of elliptical accretion discs., Aerospace Research in Bulgaria, 17, 2003, p. 17–22.
7. D i m i t r o v D. V. Thin viscous elliptical accretion discs with orbits sharing a common longitude of periastron. I. Dynamical equation for integer values of the powers in the viscosity law., Aerospace Research in Bulgaria, 19, 2006, p.16–28.
8. D i m i t r o v D. V. Thin viscous elliptical accretion discs with orbits sharing a common longitude of periastron. II. Polynomial solutions to the dynamical equation for integer values of the powers in the viscosity law., Aerospace research in Bulgaria, 21, 2007, p. 7–23.
9. D i m i t r o v D. V. Thin viscous elliptical accretion discs with orbits sharing a common longitude of periastron. III. Numerical evaluations of the validity domain of the solutions., Aerospace research in Bulgaria, 22, 2008, p. 5–23.
10. D i m i t r o v D. V. A short overview of some recent developments in the theory of non-axisymmetric accretion discs., Aerospace research in Bulgaria, 23, 2009, (in print).
11. H o w e l l S. B., D. W. H o a r d, C. B r i n k w o r t h, S. K a f k a, M. J. W a l e n t o s k y, F. M. W a l t e r, T. A. R e c t o r. "dark matter" in accretion disks., Astrophys. Journal, 685, 2008, № 1, part 1, p. 418–427.
12. M a s o n P. A., E. L. R o b I n s o n, C. L. G r a y, R. I. H y n e s. The orbital period and time-variable asymmetric accretion disk in the X–ray binary MS 1603.6 + 2600 (= UW Coronae Borealis)., Astrophys Journal, 685, 2008, № 1, part 1, p. 428–435.
13. F e r r e i r a B. T., G. I. O g i l v i e. Warp and eccentricity propagation in discs around black holes., Monthly Not. Royal Astron Soc., 392, 2009, № 1, p. 428–438.
14. C a r c i o f i A. C., J. E. B j o r k m a n. Non–LTE Monte Carlo radiative transfer. I I. Nonisothermal solutions for viscous Keplerian disks., Astrophys. Journal, 684, 2008, № 2, part 1, p. 1374–1383.

15. S h u, F, H., S. L i z a n o, D. G a l l i, M. J. C a i, S. M o h a n t y. the challenge of sub-Keplerian rotation for disk winds., Astrophys. J. Letters, 682, 2008, № 2, part 2, p. L121–L124.
16. A l p a r M. A., D. P s a l t i s. The highest dynamical frequency in the inner region of an accretion disc., Monthly Not. Royal Astron. Soc., 391, 2008, № 3, p. 1472–1476.
17. B a r r a n k o, J. A. Three–dimensional simulations of Kelvin–Helmholtz instability in settled dust layers in protoplanetary disks., Astrophys. Journal, 691, 2009, № 2, part 1, p. 907–921.
18. J a n g – C o n d e l H. Planet shadows in protoplanetary disks. I. Temperature perturbations., Astrophys. Journal, 679, 2008, № 1, part 1, p. 797–812.
19. K a t o M. T., K. N a k a m u r a, R. T a n d o k o r o, M., F u j i m o t o, S. I d a. Modifications of angular velocity by inhomogeneous magnetorotational growth in protoplanetary disks., Astrophys. Journal, 691, 2009, № 2, part 1, p. 1697–1706.
20. N e k r a s o v A. K. Electromagnetic streaming instabilities of magnetized accretion disks with strong collisional coupling of species., Astrophys. Journal, 695, 2009, № 1, part 1, p. 46–58.
21. B e c k w i t h, K., J. F. H a w l e y, J. H. K r o l i k. Where is the radiation edge in magnetized black hole accretion discs?, Monthly Not. Royal Astron. Soc., 390, 2008, № 1, p. 21–38.
22. H i r o s e S., J. H. K r o l i k, O. B l a e s. Radiation–dominated disks are thermally stable., Astrophys. Journal, 691, 2009, № 1, part 1, p. 16–31.
23. T a s s e v S. V., E. B e r s t s c h i n g e r. Kinematic density waves in accretion disks., Astrophys. Journal, 686, 2008, № 1, part 1, p. 423–431.
24. W o r k m a n J. C., P. J. A r m i t a g e. Interaction of the magnetorotational instability with hydrodynamic turbulence in accretion disks., Astrophys. Journal, 685, 2008, № 1, part 1, p. 406–414.

ТЪНКИ ВИСКОЗНИ ЕЛИПТИЧНИ АКРЕЦИОННИ ДИСКОВЕ С ОРБИТИ, ИМАЩИ ОБЩА ДЪЛЖИНА НА ПЕРИАСТРОНА. IV. ДОКАЗАТЕЛСТВО НА ХОМОГЕННОСТТА НА ДИНАМИЧНОТО УРАВНЕНИЕ, ОПРЕДЕЛЯЩО СТРУКТУРАТА НА ДИСКА, ЗА ПРОИЗВОЛНИ СТЕПЕННИ ПОКАЗАТЕЛИ *n* В ЗАКОНА ЗА ВИСКОЗИТЕТА $\eta = \beta \Sigma^n$

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Резюме

Ние разглеждаме моделите на елиптични акреционни дискове, разработени от Любаркий и др. [1], и дискутираме техните специфични свойства. В частност, ние наблягаме на възможните отклонения от

Кеплеровото въртене, магниторотационната нестабилност, външното осветяване на диска и т. н., които могат да имат място при реалните акреционни потоци (както е указано при много от наблюденията), но които не са взети под внимание в горните теоретични конструкции. Съгласно моделите, коефициентът на вискозитета η е прието да има формата на степенен закон: $\eta = \beta \Sigma^n$ (β е константа, Σ е повърхностната плътност на диска). Ние изследваме динамичното уравнение, което е получено от Любаркий и др. [1], за едно непрекъснато множество от стойности на степенния показател n. Физически приемливите значения на n обхващат областта между ≈ - 1 и ≈ + 3. Основният резултат на нашето изследване е, че динамичното уравнение, определящо структурата на елиптичните дискове, е едно ***хомогенно*** обикновено диференциално уравнение от втори ред за ***всички*** стойности на n в посочения интервал. Това е обобщение на нашия по-рано установен резултат за случая ***само на целочислени*** стойности на n.