EXOTIC ESCAPE DYNAMICS IN HÉNON-HEILES' MODEL

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Abstract
We tackle the two-body problem associated to Hénon-Heiles' renowned potential, using the qualitative analysis methods, for the limit situation of escape. The study of the infinity flow reveals many "exotic" features as regards behaviour of escape solutions.

1. INTRODUCTION

Hénon-Heiles' potential [1] was intended to model the motion of a star within a galaxy. It reads $U(q_1, q_2) = Aq_1^2 + Bq_2^2 + Cq_1^2 q_2 + Dq_2^3$, with $(q_1, q_2) \in \mathbf{R}^2$ Cartesian coordinates and $(A, B, C, D) \in (0,+\infty)^2 \times \mathbf{R}^2$.

In this paper we tackle the escape dynamics in the two-body problem associated to this potential, via qualitative analysis. Using McGehee's techniques [2, 3], we construct the *infinity manifold* and depict the flow on it. We find many "exotic" features, which point out the complexity of the escape dynamics in this field.

2. BASIC EQUATIONS AND SYMMETRIES

Consider the relative motion of a unit-mass particle w.r.t. the field source. Its dynamics is associated to the planar Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = (p_1^2 + p_2^2) / 2 - U(\mathbf{q}) , \tag{1}$$

in which $\mathbf{q} = (q_1, q_2) \in \mathbf{R}^2$ and $\mathbf{p}(= \dot{\mathbf{q}}) = (p_1, p_2) \in \mathbf{R}^2$ are the configuration vector and the momentum vector of the particle, respectively.

The system admits the first integral of energy, $H(\mathbf{q}, \mathbf{p}) = h$ (energy constant), but, given the anisotropic structure of the potential, the angular momentum is not conserved.

To tackle the escape dynamics, we pass to polar coordinates (r, θ) and polar components of velocity $(\dot{r}, r\dot{\theta})$. Then we use the McGehee-type transformations [3, 4]

$$\rho = r^{-1} \; ; \quad (x, y) = \rho^{3/2}(\dot{r}, r\dot{\theta}) \; ; \quad d\tau = \rho^{-1/2} dt \; , \tag{2}$$

which make the motion equation corresponding to (1) become

$$\begin{aligned} &\rho' = -\rho x \; , \quad \theta' = y \; , \\ &x' = -3x^2/2 + y^2 + 2(A\cos^2\theta + B\sin^2\theta)\rho + \\ &\quad + 3(C\cos^2\theta + D\sin^2\theta)\sin\theta \; , \\ &y' = -5xy/2 + 2(B - A)\rho\sin\theta\cos\theta + \\ &\quad + [3(D - C)\sin^2\theta + C]\cos\theta \; . \end{aligned} \tag{3}$$

The energy integral reads

$$\begin{aligned} &(x^2 + y^2)/2 - (A\cos^2\theta + B\sin^2\theta)\rho - \\ &- (C\cos^2\theta + D\sin^2\theta)\sin\theta = h\rho^3 \; . \end{aligned} \tag{4}$$

In (3)-(4), $' = d/d\tau$, and we kept , by abuse, the same notation for the new functions of τ .

Equations (3) have four symmetries: $S_i = S_i(\rho, \theta, x, y, \tau)$, $\quad i = \overline{0,3}$:

$$\begin{aligned} &S_0 = (\rho, \theta, x, y, \tau) = I \; (identity) \; , \quad S_1 = (\rho, \theta, -x, -y, -\tau) \; , \\ &S_2 = (\rho, \pi - \theta, -x, y, -\tau) \; , \quad S_3 = (\rho, \pi - \theta, x, -y, \tau) \; , \end{aligned} \tag{5}$$

which form an Abelian group with idempotent structure, isomorphic to Klein's group. These symmetries are of much help in describing the escape dynamics (see below).

3. INFINITY MANIFOLD AND THE FLOW ON IT

Eqs. (3) are well defined for the boundary $\rho = 0$ (escape), which is invariant to the flow, because $\rho' = 0$ for $\rho = 0$; (4) also extends smoothly to this boundary. In this way, we obtain the *infinity manifold* (pasted on the phase space), provided by (4) with $\rho = 0$:

$$M_\infty = \left\{ \begin{aligned} &(\rho, \theta, x, y) \mid \rho = 0, \; \theta \in S^1, \\ &x^2 + y^2 = 2(C\cos^2\theta + D\sin^2\theta)\sin\theta \end{aligned} \right\} ,$$

and the corresponding vector field, provided by (3) with $\rho = 0$:

$$\theta' = y, \quad x' = 5y^2/2, \quad y' = -5xy/2 + [3(D - C)\sin^2\theta + C]\cos\theta,$$

We see that the flow on M_∞ is gradientlike w.r.t. *x*. We also see that M_∞ is never defined on the whole S^1, its shape and domains of existence depending on *C* and *D*.

Let us depict the flow on M_∞. It has no physical significance, but – due to the continuity of solutions w.r.t. initial data – it yields valuable information about orbits that neighbour escape. We distinguish the following cases:

(1.1) $C = D > 0$. M_∞ is homeomorphic to a 2D-elipsoid of axis length π along θ for $\theta \in [0, \pi]$, and $M_\infty = \Phi$ else. By (7), the flow on M_∞ has two equilibria: $E^\pm(\theta, x, y) = (\pi/2, \pm\sqrt{2D}, 0)$. E^- is a source, E^+ is a sink, whereas the rest of the flow consist of heteroclinic orbits that move from E^- to E^+(Fig. 1).

(1.2) $C = D < 0$. The case is identical to (1.1), but $\theta \in [\pi, 2\pi]$ and $E^\pm(\theta, x, y) = (3\pi/2, \pm\sqrt{-2D}, 0)$.

(2.1) $0 < C \neq D > 0$. If $C \leq 3D/2$, we have the same phase portrait as (1.1). If $C > 3D/2$, M_∞ is a "dumb-bell" in $[0, \pi]$ with six equilibria: $E^-_{1,3}$ (sources), $E^+_{1,3}$ (sinks) and $E^\pm_2$ (saddles). Besides them, there are four permanent heteroclinic connections: source-saddle ($E^-_1 \to E^-_2, E^-_3 \to E^-_2$) and saddle-sink ($E^+_2 \to E^+_1, E^+_2 \to E^+_3$), and eight or four more heteroclinic orbits that form three different phase portraits (see Figs 2–4).

(2.2) $0 > C \neq D < 0$. For $C \geq 3D/2$ and $C < 3D/2$, the flows are identical to those corresponding to the cases (1.1) and (2.1), but shifted by π w.r.t. θ $(\theta \in [\pi, 2\pi])$.

(2.3) $C > 0, D < 0$. M_∞ is homeomorphic to *three* disjoint 2D-ellipsoids, spread on $[0, 2\pi]$ along θ, on which the flow is identical to the case (1.1).

(2.4) $C < 0, D > 0$. From a qualitative standpoint, this case is identical to (2.3), only the position of the ellipsoids along the θ-axis differs.

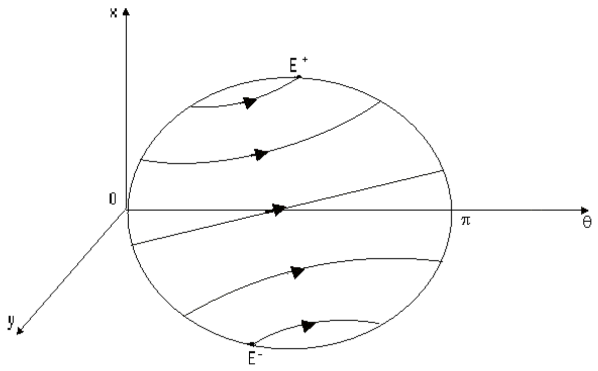

Fig. 1

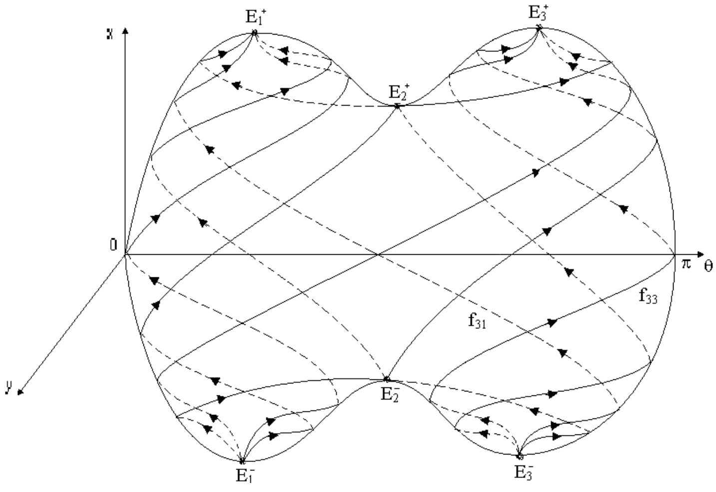

Fig. 2

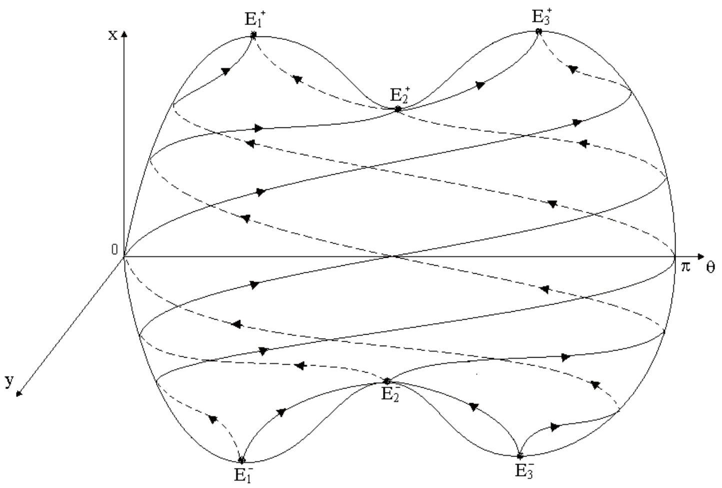

Fig. 3

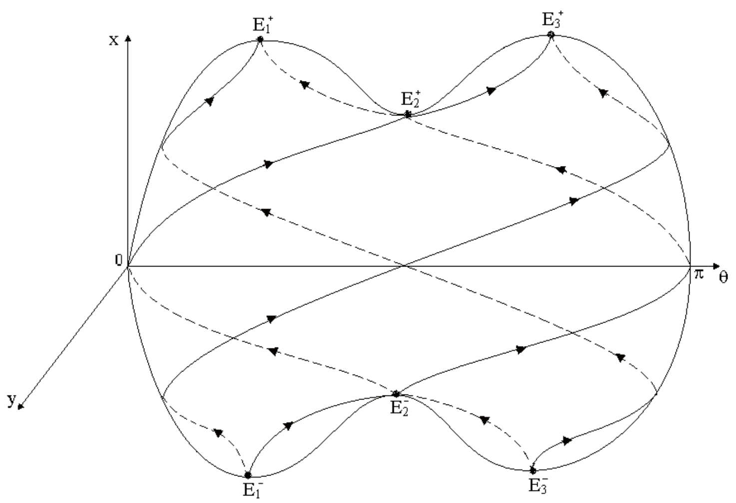

Fig. 4

4. CONCLUSIONS

Without proof, we state the main conclusions:

- Escape solutions are not regularizable.
- Escape solutions are of three different kinds: radial (with zero angular momentum), spiral (with nonzero angular momentum keeping its sign), or oscillatory (with the angular momentum alternating its sign).
- The nonradial escape motion tends asymptotically to rectilinear escape.
- The flow on the infinity manifold obeys three quite different scenarios, which are noncontradictory as regards uniqueness and symmetries. This means that there are hidden bifurcations we are not able to detect yet. However, following [4], we conjecture that saddle-saddle connections (Figs 3 and 4) are much more improbable (from the standpoint of the Lebesgue measure) than the connections illustrated in Fig. 2.

This offers a wide understanding of escape dynamics in Hénon-Heiles' model.

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