Bulgarian Academy of Sciences. Space Research Institute.
Aerospace Research in Bulgaria. 19, 2005, Sofia

# "OSCILLATOR-WAVE" MODEL AS AN INHOMOGENEOUSLY DRIVEN DYNAMICAL SYSTEM

*Vladimir Damgov*, Nikolay Erokhin**, Plamen. Trenchev**

** Space Research Institute, Bulgarian Academy of Sciences*
*** Space Research Institute, Russian Academy of Sciences*

***Abstract***

*A generalized model of an oscillator is considered, subject to the influence of external waves. It is shown that the systems of diverse physical background, encompassed by this model, should belong by their nature to the broader class of "kick-excited self-adaptive dynamical systems". The theoretical treatment includes an analytic approach to the conditions for emergence of small and large amplitudes, i.e. weak and strong non-linearity of the system.*

*The article also considers the presence of a small horseshoe in the dynamics of a particle under the action of two waves. Originally, the problem comes from the plasma physics despite the existence of some other applications of the differential equation studied here.*

## 1. Introduction

Here, the generalized "oscillator-wave" model is considered and it is shown that the inhomogeneous external influence is realized naturally and does not require any specific conditions. The systems covered by the "oscillation-wave" model immanently belong to the generalized class of kick-excited self-adaptive dynamical systems [1-5]. Attempting maximal clarity of the sequence of presentation, we consider the excitation of oscillations in a non-linear oscillator of the "pendulum" type under the influence of an incoming (fall) wave. We will show that, under certain condition, non-attenuated oscillations will arise in the system with a frequency close to the system's natural frequency and amplitude belonging to a defined discrete spectrum of possible amplitudes. A second important quality also appears - self-adaptive stability of the excited oscillations with given amplitude for a where $\Omega$ is the frequency of free oscillations having an amplitude $a$,

$$\Omega^2 = \frac{2J_1(a)}{a}\omega_o^2 \cong \omega_o^2\left[1-\frac{a^2}{8}+O(a^4)\right],$$

$J_n(\cdot)$ are Bessel functions of the first kind, $F_o = \frac{Eq}{m}$.

The excitation of continuous oscillations with frequency $\omega = \omega_s$ close to the oscillator's natural frequency is only possible under the condition $\frac{2\pi}{\lambda}a > 1$, where $\lambda$ is the wavelength of the influencing wave. As result of the interaction of the oscillator with the wave a frequency components appears in the force spectrum that is close to its natural oscillation frequency. Then the action of these spectral components becomes dominant and the right-hand side of equation (1) attains the form:

$$\frac{1}{m}Eq\sum_{n=-\infty}^{\infty}J_n(ka)\sin(vt-n\theta) =$$

$$= F_o\left\{J_{\frac{v}{\omega}-1}(ka)\sin\left[\omega t-(\frac{v}{\omega}-1)\alpha\right]-J_{\frac{v}{\omega}+1}(ka)\sin\left[\omega t+(\frac{v}{\omega}+1)\alpha\right]\right\} \tag{5}$$

Under the condition $v > \omega$ the resonance area of the nonlinear oscillator can be entered by several spectral components of the exciting wave each of which could excite the oscillator into stationary oscillations with amplitude belonging to a discrete sequence of possible amplitudes. For fixed parameters of the oscillator and the wave the excitation of oscillations with amplitude from the possible sequence of amplitudes is determined by the initial conditions. In accordance with relation (4), the values of the discrete sequence of stationary amplitudes can be calculated by the formula:

$$a_{s0} = \sqrt{8\left(1-\frac{v^2}{s^2\omega^2}\right)} \cong 4\sqrt{1-\frac{v}{s\omega_o}}, \quad s = 1,2,3... \tag{6}$$

Averaging the right-hand side of equations (3) and taking into account (5), we determine:

$$\left|\begin{aligned}\frac{da_s}{dt} &= -\delta_d a+\frac{F_o}{2\omega_s}[J_{s-1}(ka_s)+J_{s+1}(ka_s)]\sin(pt-\gamma_s)\\ \frac{d\alpha_s}{dt} &= \frac{\Omega^2-\omega_s^2}{2\omega_s}-\frac{F_o}{2\omega_s a_s}[J_{s-1}(ka_s)-J_{s+1}(ka_s)]\sin(pt-\gamma_s)\end{aligned}\right., \tag{7}$$

broad range of the incoming wave's intensity.

Leaving the details, the equation describing the motion of one particle in two electrostatic waves allows perturbation methods to be applied in its study. There are three main types of behavior in the phase space - a limit cycle, formation of a non-trivial bounded attracting set and escape to infinity of the solutions. One of the goals is to determine the basins of attraction and to present a relevant bifurcation diagram for the transitions between different types of motion.

### 2. Model of the interaction of an oscillator with an electromagnetic wave: approach applicable for small amplitudes of the system's oscillations.

Let us consider the interaction of an electromagnetic wave with a weakly dissipative nonlinear oscillator. Let the electric charge $q$ having mass $m$ oscillate along the $x$-axis under the influence of a non-linear returning force around a certain fixed point. The electromagnetic wave also propagates along the $x$-axis and has a longitudinal electric field component $E$. The equation of motion for the charge interacting with the wave can be represented as:

$$m(\ddot{x} + 2\delta_d \dot{x} + \omega_o^2 \sin x) = Eq\sin(\nu t - kx) \tag{1}$$

where $2\delta_d$ is the damping coefficient, $\omega_0$ is the natural frequency of small oscillations of the charge, $\nu$ is the wave frequency and $k$ is the wave number. The case considered is: $\nu \gg \omega_0$.

We assume that the excitation of charge oscillations by the influence of the wave does not perturb significantly the symmetry of the charge's motion around its equilibrium position and the coordinate of the charge changes according to the law

$$x = a\sin\theta, \quad \theta = \omega t + \alpha, \quad a = a(t), \quad \alpha = \alpha(t) \tag{2}$$

Substituting the solution (2) in the right hand side of equation (1) we obtain:

$$Eq\sin(\nu t - ka\sin\theta) = Eq\sum_{n=-\infty}^{\infty} J_n(ka)\sin(\nu t - n\theta).$$

Letting $\dot{x} = a\omega\cos\theta, \quad \dot{a}\sin\theta + \dot{\alpha}a\cos\theta = 0$

in accordance with the Krilov-Bogolyubov-Mitropolskii method [6], we obtain to first order:

$$\left|\begin{aligned} \frac{da}{dt} &= -2\delta_d a + \frac{F_o}{\omega}\sum_{n=-\infty}^{\infty} J_n(ka)\sin(\nu t - n\theta)\cos\theta \\ \frac{d\alpha}{dt} &= \frac{\Omega^2 - \omega^2}{2\omega} - \frac{F_o}{\omega}\sum_{n=-\infty}^{\infty} J_n(ka)\sin(\nu t - n\theta)\cos\theta \end{aligned}\right. \tag{3}$$

where $p = \nu - \frac{\nu}{\omega_s} << \omega_s$, $\gamma_s = \frac{\nu}{\omega_s}\alpha_s$ $\omega_s = \frac{\nu}{s}$.

In accordance with the familiar recurrence relations for Bessel functions, equations (7) can be represented in the form:

$$\begin{cases} \frac{da_s}{dt} = -\delta_d a_s - \frac{\nu F_o}{\omega_s^2 k a_s} J_s(ka_s)\sin(pt - \gamma_s) \\ \frac{d\alpha_s}{dt} = \frac{\Omega_s^2 - \omega_s^2}{2\omega_s} - \frac{F_o}{\omega_s a_s} J'_s(ka_s)\cos(pt - \gamma_s) \end{cases} \quad (8)$$

In the case of stationary oscillations ($\frac{da_s}{dt} = 0$ and $\frac{d\alpha_s}{dt} = 0$) from equations (8) we find:

$$tg\gamma_s = \frac{2\delta_d a_s \omega_s^2 k}{(\Omega_s^2 - \omega_s^2)\nu} \frac{J'_s(ka_s)}{J_s(ka_s)}$$

The connection between the intensity of the wave's longitudinal component and the amplitude of oscillations has the form:

$$F_o^2 = \left[\frac{a_{so}^2 \omega_s^2 \delta_d k}{\nu J_s(ka_{so})}\right]^2 + \left[\frac{a_{so}(\Omega_s^2 - \omega_s^2)}{2J'_s(ka_{so})}\right]^2. \quad (9)$$

For high intensities of the wave, equation (9) can be represented as:

$$F_o = \frac{a_{so}^2 \omega_s^2 (a_s - a_{so})}{8J'_s(ka_s)}.$$

The first term in formula (9) represents the minimal threshold value $F_o$ of the wave's intensity. If the intensity of the wave is smaller than this threshold value only the excitation of forced oscillations with frequency equal to the wave's frequency is possible. For wave intensities above the threshold value depending on the initial conditions, the oscillator's motion is realized with one of the amplitudes from the discrete sequence (6). When $\nu > \omega_0$ each amplitude is realized for oscillation frequency close to the oscillator's natural frequency. Using the approach, developed in [3], it is not difficult to show that for fixed values of the frequency ν and the amplitude $F_o$ of the external force the oscillator's motion with amplitude from the discrete sequence (6) is stable.

The performed analysis shows that the continuous wave having a frequency much larger than the frequency of a given oscillator can excite in it oscillations with a frequency close to its natural frequency and an amplitude belonging to a discrete set of possible stable amplitudes.

The settling of certain particular amplitude depends on the initial conditions. When the motion becomes stationary the amplitude's value practically does not depend on the wave's intensity when the latter changes over a significant range above a certain threshold value. This is reminiscent of Einstein's explanation of the photoelectric effect using Planck's quantization hypothesis. In this case the absorption is also independent of the incoming wave's intensity. Besides, the absorbed frequencies can be expressed as integer multiples of a certain basic frequency reminding of resonance phenomena.

### 3. Approach for large amplitudes of the oscillations in a nonlinear dynamical system subjected to the influence of a wave

Let the nonlinear oscillator is an electric charge $q$ with mass $m$ and it is able to oscillate along the X-axis with a small friction force $2\delta_o \dot{X}$. Let an electromagnetic wave propagating along the X-axis acts upon the oscillating charge. Let us assume that the wave has a longitudinal component of the electric field $E_X$. The equation of the charge motion becomes

$$\ddot{X} + 2\delta_o \dot{X} + \omega_o^2 \sin X = P_o \sin(\nu t_r - kX - \varphi), \qquad (10)$$

where $\omega_o$ is the resonant frequency of small amplitude oscillations $P_o = E_X q / m$; $\nu$, $\varphi$ and $k$ are the frequency, the initial phase and the wave number respectively, $t_r$ is the real time. We assume $\nu >> \omega_o$.

Let us introduce the dimensionless time $t = \omega_o t_r$. In this case, Eq. (10) takes the form

$$\ddot{X} + 2\delta_d \dot{X} + \sin X = F_o \sin(\frac{\nu}{\omega_o} t - kX - \varphi),^{*)} \qquad (11)$$

where $2\delta_d = 2\delta_o / \omega_o$, $F_o = P_o / \omega_o^2$

In order to integrate the Eq. (11) with the methods of the Theory of nonlinear oscillations, we apply the approach developed in [3, 7]. We introduce a new variable $y$ and nonlinear time $\tau$,

$$y = \operatorname{sign} x \sqrt{2 \int_0^x \sin x' dx'} = 2 \sin \frac{x}{2}, \qquad (12)$$

*) Similar equations describe the behavior of cosmic charged particles in certain conditions, the processes in radio-frequency driven, quantum-mechanical Josephson junctions, charge density wave transport and other systems.

$$\frac{dt}{d\tau}=\frac{dX}{dy}=\frac{y}{\sin[X(y)]}=G(y) \tag{13}$$

So, the nonlinear reactive term $\sin X$ in the Eq. (11) may be "excluded". The functions $X(y)$ and $G(y)$ in (13) are easily expressed by taking into account (12) in the form

$$X(y)=2\arcsin\left(\frac{y}{2}\right),\quad G(y)=\frac{1}{\sqrt{1-\frac{y^2}{4}}} \tag{14}$$

Substituting (12) and (13) in (11) we obtain

$$\frac{d^2y}{d\tau^2}+y=\left\{-2\delta_d\frac{dy}{d\tau}+F_o\sin\left[\frac{\nu}{\omega_o}t(\tau)-kX(y)-\varphi\right]\right\}G(y) \tag{15}$$

The further consideration will be performed for the following interval of $y$ values: $-2<y<2$.

Before the integration of the Eq. (15) we will mention, that the solution will be quasi-harmonic with nominal frequency $\omega_n=\nu/N$, where $N\gg1$ is a positive odd number, however $\omega_n\sim\omega_o$. That is why we will write the Eq. (15) in the following form:

$$\frac{d^2y}{d\tau^2}+\beta^2y=-2\delta_d\frac{dy}{d\tau}+F_o\sin\left[\frac{\nu}{\omega_o}t(\tau)-kX(y)-\varphi\right]G(y)+(\beta^2-1)y, \tag{16}$$

where $\beta\sim1$ corresponds to the difference from the resonant frequency.

We assume that in excitation of the charge oscillations by the wave its motion is symmetric with respect to the equilibrium and the charge coordinate changes in agreement with

$$y=R\cos\beta\tau=R\cos\psi \tag{17}$$

The dependence of the normalized time $t$ on the angle $\psi$ can be expressed in agreement with (13), (14) and (17) as

$$t=\frac{1}{\beta}\int_0^{\psi}\frac{d\Psi}{\sqrt{1-\frac{R^2}{4}\cos^2\Psi}}.$$

The normalized period of the oscillations is

$$T_o=\frac{1}{\beta}\int_0^{2\pi}\frac{d\Psi}{\sqrt{1-\frac{R^2}{4}\cos^2\Psi}}=\frac{4}{\beta}K\left(\frac{R}{2}\right), \tag{18}$$

where $K\left(\frac{R}{2}\right)$ is a complete elliptic integral of first kind.

By use of (18) the coefficient β is expressed in the form

$$\beta = \frac{2\nu K\left(\frac{R}{2}\right)}{\pi\omega_o N}.$$

Now we can solve Equation (16).

The shortened (reduced) differential equations for the amplitude R and phase φ may be written as:

$$\frac{dR}{d\tau} = -\frac{1}{2\pi\beta}\int_0^{2\pi} L[R\cos\psi, -\beta R\sin\psi, e(\psi-\varphi)]\sin\psi d\psi, \qquad (19)$$

$$\frac{d\varphi}{d\tau} = -\frac{1}{2\pi\beta}\int_0^{2\pi} L[R\cos\psi, -\beta R\sin\psi, e(\psi-\varphi)]\cos\psi d\psi, \qquad (20)$$

where

$$L[R\cos\psi, -\beta R\sin\psi, e(\psi-\varphi)] = 2\delta_d\beta R\sin\psi + \frac{F_o}{\sqrt{1-R^2/4\cos^2\psi}}\sin\left[\frac{\pi N}{2K(R/2)}\right]$$

Let us introduce the following designations:

$$\begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} = \int_0^{K(R/2)} \sin(rZ)\cos[D(Z)]\begin{Bmatrix} \mathrm{sn}Z \\ \mathrm{cn}Z \end{Bmatrix} dz, \qquad (21)$$

$$\begin{Bmatrix} H_3 \\ H4 \end{Bmatrix} = \int_0^{K(R/2)} \cos(rZ)\sin[D(Z)]\begin{Bmatrix} \mathrm{sn}Z \\ \mathrm{cn}Z \end{Bmatrix} dZ \qquad (22)$$

$$\begin{Bmatrix} H_5 \\ H6 \end{Bmatrix} = \int_0^{K(R/2)} \cos(rZ)\cos[D(Z)]\begin{Bmatrix} \mathrm{sn}Z \\ \mathrm{cn}Z \end{Bmatrix} dZ \qquad (23)$$

$$\begin{Bmatrix} H_7 \\ H8 \end{Bmatrix} = \int_0^{K(R/2)} \sin(rZ)\sin[D(Z)]\begin{Bmatrix} \mathrm{sn}Z \\ \mathrm{cn}Z \end{Bmatrix} dZ \qquad (24)$$

where $Z = F(\psi, R/2)$ is an incomplete elliptic integral of the first kind,

$D(Z) = 2kE\left[\arcsin\left(\frac{R}{2}cnZ\right), \frac{2}{R}\right]$, $E[\cdot,\cdot]$ is an incomplete elliptic integral of second kind, sn$Z$ and cn$Z$ are sine and cosine of the amplitude (the Jacobi elliptic functions),

$$r=\frac{\pi N}{2K(R/2)}.$$

Taking into account the expressions (21)-(24), the shortened Equations (19) and (20) for establishing the amplitude R and phase $\varphi$ take the form

$$\frac{dR}{d\tau}=-\delta_d R-\frac{F_o}{2\pi\beta}\left[(H_1-H_3)\cos\varphi-(H_5+H_7)\sin\varphi\right],$$

$$\frac{d\varphi}{d\tau}=-\frac{F_o}{2\pi\beta R}\left[(H_2-H_4)\cos\varphi-(H_6+H_8)\sin\varphi\right]-(\beta-1)..$$

For the stationary mode ($\frac{dR}{d\tau}=0,\ \frac{d\varphi}{d\tau}=0$) we obtain the following expressions for the established values of the amplitude R and the phase $\varphi$:

$$R=\frac{F_o\left[(H_1-H_3)(H_6-H_8)-(H_2-H_4)(H_5+H_7)\right]}{2\pi\beta\delta\sqrt{\left[\sigma(H_1-H_3)-(H_2-H_4)\right]^2+\left[\sigma(H_5+H_7)-(H_6+H_8)\right]^2}} \qquad (25)$$

$$\varphi=arctg\,\frac{\sigma(H_1-H_3)-(H_2-H_4)}{\sigma(H_5+H_7)-(H_6+H_8)} \qquad (26)$$

where $\sigma=(\beta-1)/\delta_d$.

**4. Presence of a small horseshoe in the dynamics of a particle under the action of two waves**

Originally the problem comes from plasma physics [8] even though that there exist and some other applications of the differential equation which we shall study. Leaving the details, the equation which describes the motion of one particle in two electrostatic waves is given by

$$\ddot{x}=-M\sin x-P\sin(x-t), \qquad (27)$$

where $x$ is the position of the particle measured in the frame of one of the waves, $P$ and $M$ are dimensionless amplitudes of the waves. We shall extend our model introducing a damping term in (27), and we shall also assume that $P \ll M$. Under these assumptions the equation that governs the motion of the particle can be written as

$$\ddot{x}+\varepsilon\delta\dot{x}+\sin x=\varepsilon f\sin(vt-x), \qquad (28)$$

where $x$ is again the position of the particle measured in the frame of one of the waves, whereas $\delta$, $f$ and $v$ are real non-negative constants. The form of the equation allows perturbation methods to be applied in its study. Our preliminary numerical investigation of (28) revealed very rich dynamics depending on the change of parameters and initial conditions. There are three main types of behaviour in the phase space of (28) that can be observed:

- Approaching a limit cycle;
- Formation of a non-trivial bounded attracting set;
- Escape to infinity of the solutions of (28).

One question which is of natural interest here, is to determine the basins of attraction, and to present a relevant bifurcation diagram for the transitions between different types of motion. Although that there has been reached a significant progress in the understanding of the behavior of driven non-linear oscillators, there exist some obstacles that prevent clarifying the dynamics of particular examples. In our work we present a rigorous result for existence of horseshoe-like dynamics for (28) and hence for exhibiting the phenomena deterministic chaos. Our result is as follows:

**Theorem 1.** *The sufficient conditions for transition to chaotic motion in the dynamics of equality* (28) *is fulfillment of*

$$4\delta < f\pi v^2\left(\frac{1}{\cosh(\pi v/2)}-\frac{1}{\sinh(\pi v/2)}\right), \tag{29}$$

or

$$4\delta < f\pi v^2\left(\frac{1}{\cosh(\pi v/2)}+\frac{1}{\sinh(\pi v/2)}\right). \tag{30}$$

The organization of our study is as follows: In next section we shall give a short account of the Melnikov method in form convenient for our problem. Then we shall prove Theorem 1. In the last section we shall say some words on the physical implication of our result.

### 4.1. Short summary of the Melnikov method

We shall explain the Melnikov technique following [9].

#### A. General assumptions and geometric structure of the non-perturbed system.

Consider the system of differential equations

$$\dot{x} = JD_xH(x) + \varepsilon g(x,t,\varepsilon) \tag{31}$$

where $(x,t) \in R^2 \times T^1$ and $J$ is the symplectic matrix defined by $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

We get the following structural assumptions:

1. The functions

$$JD_x H : R^2 \to R^2$$

$$g : R^2 \times R \times R \to R^2,$$

are defined and at least $C^2$ differentiable on their respective domains of definition, and that $g$ is periodic in $t$ with period $T = 2\pi / \omega$.

2. The system (31) with $\varepsilon = 0$ is referred as *unperturbed system*. About it we shall assume that it possesses a hyperbolic fixed point $x_{o,h}$ connected to itself by a homoclinic orbit $x_h(t) \equiv (x_h^1(t), x_h^2(t))$.

3. Let denote by $W^s(x_{0,h})$ the set of points $x \in R^2$ that approach $x_{0,h}$ as $t \to \infty$, and by $W^u(x_{0,h})$ the set of points $x \in R^2$ that approach $x_{0,h}$ as $t \to -\infty$, under the action of the unperturbed flow

$$\dot{x} = JD_x H(x) \qquad (32)$$

$W^s(x_{0,h})$ is referred as *asymptotically stable manifold* for $x_{0,h}$, and $W^u(x_{0,h})$ is referred as *asymptotically unstable manifold* for $x_{0,h}$. Denote by $\Gamma_{x_{o,h}} \equiv \{x \in R^2 \mid x = x_h(t), t \in R\} \cup \{x_{0,h}\} = W^s(x_{0,h}) \cap W^u(x_{0,h}) \cup \{x_{0,h}\}$. We shall assume that interior of $\Gamma_{x_{o,h}}$ is filled with continuous family of periodic orbits $x^\alpha(t)$ of (32) with period $T^\alpha$, $\alpha \in [-1, 0]$ and $\lim_{\alpha \to 0} x^\alpha(t) = x_h(t)$ and $\lim_{\alpha \to 0} T^\alpha = \infty$.

When viewed in three-dimensional space $R^2 \times S$, the hyperbolic fixed point $x_{0,h}$ turns to hyperbolic periodic orbit $\gamma(t) \equiv (x_{0,h}, \theta(t) = \omega t + \omega_o)$ of the system

$$\begin{aligned} \dot{x} &= JD_x H(x) \\ \dot{\theta} &= \omega \end{aligned} \qquad (33)$$

and so do $W^s(x_{0,h})$ and $W^u(x_{0,h})$ which turn to two-dimensional asymptotic manifolds $W^s(\gamma(t))$ and $W^u(\gamma(t))$ which coincide along the two-dimensional homoclinic manifold

$$\Gamma_{\gamma(t)} \equiv \{(x,\theta)\in R^2\times S \mid x = x_h(t), t\in R\} \cup \{x_{0,h}\times S\}.$$

### B. Geometric structure of the perturbed phase space.

Here we shall argue that most of the upper structure goes over for the perturbed system.

$$\begin{aligned}\dot{x} &= JD_x H(x) + \varepsilon g(x,\theta,\varepsilon)\\ \dot{\theta} &= \omega\end{aligned} \tag{34}$$

**Proposition 1**. *For $\varepsilon$ sufficiently small the periodic orbit $\gamma(t)$ of* (7) *survives as a periodic orbit, $\gamma_\varepsilon(t)=\gamma(t)+O(\varepsilon)$, of* (34) *having the same stability type as $\gamma(t)$, and depending on $\varepsilon$ in a $C^2$ manner. Moreover, the local stable and unstable manifolds $W^s_{loc}(\gamma_\varepsilon(t))$ and $W^u_{loc}(\gamma_\varepsilon(t))$ of $\gamma_\varepsilon(t)$ remain also $C^2$ $\varepsilon$-close to the local stable and unstable manifolds $W^s_{loc}(\gamma(t))$ and $W^u_{loc}(\gamma(t))$ of $\gamma(t)$, respectively.*

**Remark 1.** *The concept for local stable and unstable manifolds becomes clear when one represents the stable and unstable manifolds of the hyperbolic fixed point (periodic orbit) locally. For details see* [9] *or* [10].

Now, the global stable and unstable manifolds of $\gamma_\varepsilon(t)$ are

$$W^s(\gamma_\varepsilon(t)) = \bigcup_{t\le 0}(x,\theta)^t\left(W^s_{loc}(\gamma_\varepsilon(t))\right),$$
$$W^u(\gamma_\varepsilon(t)) = \bigcup_{t\ge 0}(x,\theta)^t\left(W^u_{loc}(\gamma_\varepsilon(t))\right)$$

where we denote by $(x,\theta)^t$ the phase flow of (34).

Consider the following cross-section of the plane $R^2\times S$

$$\Theta^{\theta_o} = \{(x,\theta)\in R^2 \mid \theta=\theta_o\}.$$

$\Theta^{\theta_o}$ is parallel to the $x$-plane (and coincides with the $x$-plane for $\theta_o=0$). Note that $\gamma(t)\cap\Theta^{\theta_o}=x_{o,h}$ and $\Gamma_\gamma\cap\Theta^{\theta_o}=\{x\in R^2 \mid x=x_{o,h}, t\in R\}=\Gamma_{x_{o,h}}$. Consider a trajectory

$$\left(x_\varepsilon(t),\theta(t)\right), \tag{35}$$

of the perturbed vector field (34). Then its projection onto $\Theta^{\theta_o}$ is given by $\left(x_\varepsilon(t),\theta_o\right)=W^s\left(\gamma_\varepsilon(t)\right)\cap W^u\left(\gamma_\varepsilon(t)\right)$. Since $x_\varepsilon(t)$ actually depends on $\theta_o$ (as opposed to $x(t)$, for some solutions ($x_\varepsilon(t),\theta(t)$ of (34)), the perturbed vector field (34) is non-autonomous, which may result in a very complicated picture of (35) in $\Theta^{\theta_o}$, possibly intersecting itself. The points from the Poincare map $P_\varepsilon$ defined as the successive points of intersection of the solution $(x_\varepsilon(t), \theta(t))$ with $\Theta^{\theta_o}$, will be mapped also onto this curve. It turns out that these points can form very complicated (non-trivial) sets due to transversal intersection of the asymptotic manifolds $W^s(\gamma_\varepsilon(t))$ and $W^u(\gamma_\varepsilon(t))$. One computable criterion that assures such dynamics is given by:

**Proposition 2.** [19] *Suppose that we have a point* $(t_o, \theta_o) = (\bar{t}_o, \bar{\theta}_o)$ *such that*

1. $M(\bar{t}_o, \bar{\theta}_o) = 0$,
2. $\left.\dfrac{\partial M}{\partial t_o}\right|_{(\bar{t}_o, \bar{\theta}_o)} \neq 0$,

*where* $M(t_o, \theta_o)$ is *the Melnikov vector*

$$M(t_o, \theta_o) = \int_{-\infty}^{\infty} DH(x_h(t)).g(x_h(t), \omega t + \theta_o, 0)dt$$

*Then* $W^s(\gamma_\varepsilon(t))$ *and* $W^u(\gamma_\varepsilon(t))$ *intersect transversely at* $(x_h(-\bar{t}_o) + O(\varepsilon), \bar{\theta}_o)$ *and consequently (from the Smale-Birkhoff homoclinic theorem) for the map* $P_\varepsilon$ *there exists an integer* $n > 1$ *that* $P_\varepsilon^n$ *has an invariant Cantor set on which it is topologically conjugate to a full shift of N symbols.*

## C. Proof of Theorem 1.

Consider equation (28) written in the form

$$\begin{aligned} \dot{x}^1 &= x^2, \\ \dot{x}^2 &= \sin x^1 + \varepsilon[-\delta x^2 + f\sin(\theta - x^1)], \\ \dot{\theta} &= \nu. \end{aligned} \tag{36}$$

Then the following lemma holds:

**Lemma 1.** *For* $\varepsilon = 0$ *system (36) contains hyperbolic periodic orbit*

$$M = (\bar{x}^1, \bar{x}^2, \theta(t)) = (\pm\pi, 0, \nu t + \theta_o) \in R^2 \times T^1.$$

*This orbit is connected to itself by a pair of 2-dimensional homoclinic manifolds given by*

$$\left(x_{\pm}^{1}(t), x_{\pm}^{2}(t), \theta(t)\right)=\left(\pm 2\arcsin(\tanh(t-t_o)), \pm\frac{2}{\cosh(t-t_o)}, vt+\theta_o\right). \quad (37)$$

***Proof.*** We easily see that $(\pm\pi, 0)$ is a hyperbolic fixed point of $\dot{x}^1 = x^2$, $\dot{x}^2 = \sin x^1$, linearizing (36) (for ε = 0) about it. A trivial check gives that (for ε = 0) (37) in solution of (36). Furthermore, using asymptotic of (37) we obtain that it connects $(\pm\pi, 0, vt+\theta_o)$ to itself. This proves the lemma.

Using Proposition 2 and hyperbolicity of (37), we conclude that for ε ≠ 0, (37) turns to hyperbolic periodic, orbit which we shall shortly denote by $\gamma_{\varepsilon,+}(t) \equiv (\pm\pi + O(\varepsilon), 0 + O(\varepsilon), vt + \theta_o)$. From Proposition 2 it follows that its asymptotic manifolds $W^s(\gamma_{\varepsilon,+}(t))$ and $W^u(\gamma_{\varepsilon,+}(t))$ will intersect transversely if the corresponding Melnikov vector

$$M_{\pm}(t_o, \theta_o, \delta, f, v) =$$

$$= \int_{-\infty}^{\infty}\left[-\delta\left(x_{h,\pm}^{2}(t-t_o)\right)^2 + f\sin\left(vt+\theta_o - x_{h,\pm}^{1}(t-t_o)\right)x_{h,\pm}^{2}(t-t_o)\right]dt =$$

$$= \int_{-\infty}^{\infty}\left[-\delta\left(\frac{\pm 2}{\cosh(t-t_o)}\right)^2 + f\left(\frac{\pm 2}{\cosh(t-t_o)}\right)\sin(vt+\theta_o \pm 2\arcsin(\tanh(t-t_o)))\right]dt,$$

has a simple zero. Furthermore we fix $\theta = \theta_o$, which defines the cross-section

$$\Theta^{\theta_o} = \{(x_1, x_2, \theta) \mid \theta = \theta_o \in [0, 2\pi)\},$$

and consider the Poincare map $P_{\varepsilon}^{\theta_o}: \Theta^{\theta_o} \to \Theta^{\theta_o}$ generated by the flow (36). In order to make the conclusions we pursue about the dynamics of $P_{\varepsilon}^{\theta_o}$ we need to compute $M_{\pm}(t_o, \theta_o, \delta, f, v)$. After some algebra we obtain for $M_{\pm}$

$$M_{\pm}(t_o, \theta_o, \delta, f, v) = -8\delta \pm 2f\sin(vt_o + \theta_o)\ [I_1 \mp 2I_2],$$

where

$$I_1 = \int_{-\infty}^{\infty}\frac{1-\sinh^2\tau}{\cosh^3\tau}\cos(v\tau)d\tau = v\int_{-\infty}^{\infty}\frac{\sinh\tau}{\cosh^2\tau}\sin(v\tau)d\tau,$$

$$I_2 = \int_{-\infty}^{\infty}\frac{\sinh\tau}{\cosh^3\tau}\sin(v\tau)d\tau = \frac{v}{2}\int_{-\infty}^{\infty}\frac{1\sinh\tau}{\cosh^2\tau}\cos(v\tau)d\tau.$$

The integrals $I_1$ and $I_2$ are evaluated by the methods of residues. The standard calculation gives

$$I_1 = \frac{\pi v^2}{\cosh(\pi v/2)}; \quad \text{and} \quad I_2 = \frac{\pi v^2}{2\sinh(\pi v/2)}.$$

Hence, for the Melnikov vector $M_\pm$ we obtain

$$M_\pm(t_o,\theta_o,\delta,f,v) = -8\delta \pm 2f\pi v^2\left[\frac{1}{\cosh(\pi v/2)} \mp \frac{1}{\sinh(\pi v/2)}\right]\sin(vt_o+\theta_o) \tag{38}$$

Then fulfillment of (3) assures existence of simple zero for

$$M_+(t_o,\theta_o,\delta,f,v) = 0,$$

and hence transversal intersection of the asymptotically stable manifold $W^s(\gamma_{\varepsilon,+})$ and asymptotically unstable manifold $W^u(\gamma_{\varepsilon,+})$, whereas the fulfillment of (30) assures existence of simple zero for

$$M_-(t_o,\theta_o,\delta,f,v) = 0$$

and hence transversal intersection of $W^s(\gamma_{\varepsilon,-})$ and $W^u(\gamma_{\varepsilon,-})$. Now, from Proposition 2 it follows for ε > 0 sufficiently small there exists an integer $n > 1$ such that the map $P_\varepsilon^{\theta_o}$ has an invariant Cantor set, subset of the Poincare section $\Theta^{\theta_o}$, on which the power ($P_\varepsilon^{\theta_o}$) is conjugate to a full shift of $N$ symbols.

The last implies that high sensitiveness of solution to the choice of initial conditions, or equivalently deterministic chaos.

## 5. Conclusion

The analysis shows the following two essential features of the system considered.

1. There exists a discrete set of possible stationary stable amplitudes, which can be approximately determined under certain conditions.
2. There exists a threshold value for the amplitude such that for values above it the discrete states are stable.

The phenomenon of continuous oscillation excitation with amplitude belonging to a discrete set of stationary amplitudes has been demonstrated on the basis of a common model – an oscillator under wave influence. It is shown that the conditions necessary for the manifestation of this phenomenon are realized in a natural way in an oscillator system interacting with a continuous electromagnetic wave.

Modeling the system of an oscillating charge under wave influence has been considered. It has been shown that the continuous wave with spectral components, considerably higher than the oscillating charge's natural frequency, excites charge oscillations with a quasi-natural frequency and amplitude belonging to a discrete set of the possible stationary amplitudes, depending only on the initial conditions. The considered model may be used for phenomenological investigation of plasma particles with electromagnetic waves interactions and waves in the Earth ionosphere and planetary magnetospheres.

In fact the main consequence of Theorem 1 is the strong dependence of the solution of (28) on the choice of initial conditions. The phenomenon deterministic chaos arises often in the dynamics of the driven non-linear oscillators. In this regard our result is not surprising. Anyway, we think that it is useful to present such a condition for the parameters of the system which guarantees appearance of a Smale horseshoe like dynamics, since usually the homoclinic bifurcation (due to a simple zero of the Melnikov vector) is one of the first bifurcations that occur in the transition from regular to irregular motion for **a** given system. We want to emphasis that the homoclinic tangency (predicted with a good accuracy by the Melnikov analysis), as a rule, implies formation of a fractal boundary for the basins of attraction. The last makes difficult clarifying the global dynamics on specific examples. The other two types of motion, outlined in section 4, are treated by the means of the averaging theory using a sub-harmonic Melnikov function. Our results are subject to a forthcoming paper.

### References


1. D a m g o v, V. and I.Popov. "Discrete" Oscillations and Multiple Attractors in Kick-Excited Systems. *Discrete Dynamics in Nature and Society* (*An International Multidisciplinary research and Review Journal*) 3(No.6), 1-26 (1999)
2. D a m g o v, V.. "Quantized" Oscillations and Irregular Behaviour of a Class of Kick-Excited Self-Adaptive Dynamical Systems. *Progress of Theoretical Physics Supplement, Kyoto, Japan* 3(No.5), 41-47 (2000)
3. D a m g o v, V.. Modeling Systems and Mechanisms of Oscillation Excitation. *Earth, Moon and Planets* 61, 87-117 (1993)
4. D a m g o v, V. and I.Popov. Multiple Attractors in a Class of Dynamical Systems. *Comptes Rendus de la Academy Bulgare des Science* 52(No.5-6), 31-34 (1999)
5. D a m g o v, V. and P.Georgiev. General Conditions for Resonance Oscillations Excitation Under the Action of External Nonlinear Force. *Comptes Rendus de la Academy Bulgare des Science* 52(No.9-10), 22-25 (1999)
6. B o g o l j u b o v, N. and Y.A. Mitropolsky. *Asimptotical Methods in the Theory of Nonlinear Oscillations*. Nauka Publ., Moskow (1916)

7. D a m g o v, V.. *Nonlinear and Parametric Phenomena in Radiophysical Systems.* "Prof. M.Drinov" Academic Publ. House, Sofia (2000)
8. E s c a n d e, D. Large-Scale Stochasticity in Hamilt.onian Systems. *Pliysica Scripta,* T2/1 (1982), 126-141
9. W i g g i n s, S. Introduction to Applied Nonlinear Dynamics and Chaos, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, 1989
10. W i g g i n s, S. Global Bifurcations and Chaos, Springer-Verlag, New York, Berlin, Heidelberg, 1988

# ОСЦИЛАЦИОННО-ВЪЛНОВ МОДЕЛ КАТО НЕХОМОГЕННО ВОДЕНА ДИНАМИЧНА СИСТЕМА

*Владимир.Дамгов , Николай Ерохин , ПламенТренчев*

**Резюме**

Разгледан е обобщен модел на осцилатор, намиращ се под външно вълново въздействие. Показано е, че системи с различна физическа природа, обединявани от този модел, принадлежат към по-общия клас "кик-възбудими само-адаптивни динамични системи". Теоретичното разглеждане включва анализ в условията на големи и малки амплитуди, т.е. случаите на силна и слаба нелинейност на системата. Статията разглежда също наличието на така наречената подкова на Смейл в динамиката на частица, намираща се под въздействието на две вълни. Този проблем идва от физиката на плазмата, но диференциалните уравнения, разглеждани в статията, имат и множество други приложения.