NON-LINEAR OSCILLATOR UNDER EXTERNAL SYNCHRONIZING INFLUENCE: COMPARISON OF PERTURBATION AND CANONICAL METHODS OF ANALYSIS

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INTRODUCTION

The analysis of oscillations and vibrations is usually reduced to the problem of a non-linear oscillator, subjected to external periodic influence (perturbation). With the development of perturbation methods, two main directions have formed: canonical (Hamiltonian) methods and non-canonical (non-Hamiltonian) methods.

The development of these two main directions as well as an overview of the principal methods of analysis of perturbed nonlinear oscillator are given in [1].

The present work compares the results obtained by different perturbation methods. It is shown that the solutions using action-angle variables (obtained through canonical transformations) and those obtained using Kuzmak's method produce equivalent results.

GENERATING SOLUTION

Consider a generalized non-linear oscillator described by the following system of equations:

$$(1) \qquad \left\| \begin{aligned} &\frac{dx}{dt} - p = 0 \\ &\frac{dp}{dt} + f(x,T) = \mu F_v\left(\frac{dx}{dt}, x, t, T\right) \end{aligned} \right.$$

where $0 \le \mu << 1$ is a small parameter, T is secondary scaling /slow time/,

$T = T_o + \mu T, \quad T_o = const, \quad dT/dt = \mu.$

Let μF_v depend on t primarily through the phase /quick/ variable, such that:

$$(2) \qquad \frac{d\gamma(t)}{dt} = \Gamma(t) \qquad \frac{d\Gamma(t)}{dt} = O(\mu)$$

/the latter means that $(1/\mu)(\Gamma(t)/dt)$ is limited in the neighborhood of $\mu = 0$/

In (2) Γ is the circular frequency of external influence, $2\pi/\Gamma(t)$ is the period of external influence.

We are looking for a solution of (1), synchronous with the external influence, i.e. with $\gamma(t)$ which has circular frequency:

$$(3) \qquad \Omega(t) = \frac{m}{n}\Gamma(t) \quad m, n = 1, 2, 3, \ldots, \qquad \frac{d\Omega(t)}{dt} = O(\mu)$$

The period of the generating solution in the non-autonomous regime is:

$$\Pi_{or}(t) = \frac{2\pi}{\Omega(t)}.$$

The following substitution is made: $\frac{m}{n}\frac{d\gamma(t)}{dt} = \Omega(t) = \frac{2\pi}{\Pi_{or}(t)}$. The frequency of the oscillations in the autonomous regime is $\omega(\AA, T) = \frac{2\pi}{\ddot{I}(\AA\,\grave{\rho})}$ where E = const is an integration constant and $\Pi(\mathrm{E}, T)$ is the corresponding oscillation period.

That yields for the non-autonomous regime/in the presence of synchronizing influence/:

$$(4) \qquad \frac{d\Psi}{dt} = \Omega(t) = \xi(E,T)\omega(E,T); \qquad \Pi_{or}(t) = \frac{\Pi(E,t)}{\xi}$$

where $\xi = \begin{cases} 1, & \text{in the absence of frequency lock-on} \\ \dfrac{\Omega}{\omega}, & \text{in the presence of frequency lock-on} \end{cases}$

We represent the frequency Ω in an asymptotic series:

(5) $$\Omega(t) = \Omega_o(T) + \mu[\Omega_1(T) + W_1(t,T)] + \mu^2[\Omega_2(T) + W_2(t,T)] + \ldots$$

This means that:

(6) $$\left|\begin{array}{l} \Psi(t) = \Psi_{\hat{\imath}}(t) + \hat{I}(\mu), \\ \xi(E,t,T) = \xi_o(E,T) + \hat{I}(\mu) \\ \ddot{I}_{or}(t) = \dfrac{\ddot{I}(\mathring{A}\,\grave{\rho})}{\xi_o(E,T)} + \hat{I}(\mu) \end{array}\right.$$

Let's now consider the generating solution in non-autonomous regime by making the substitution:

(7) $\mu = 0, \quad T = T_o = const, \quad E(T) = const, \quad \Omega = const$

Then $\Omega = \Omega_o(T) = \xi_o(E,T)\omega(E,T) = const$ and the solution of the system (1) is sought to be in the form:

(8) $$\left|\begin{array}{l} x = x_d[E, t + t_o, T] = x_a[E, (t + t_o)\xi(E,T), T] \\ p = p_d[E, t + t_o, T] = p_a[E, (t + t_o)\xi(E,T), T] \end{array}\right.$$

where x_A and p_A are solutions of the system /1/ in the autonomous case [1].

$E = \dfrac{1}{2}p^2 + V(x,T)$, $V(x,T) = \int_0^x f(x',T)dx'$ is the potential energy.

The functions /8/ satisfy the generating system when condition (7) is satisfied, i.e.

(9) $$\left|\begin{array}{l} \dfrac{dx}{dt} = \xi(E,T)p \\ \dfrac{dp}{dt} = -\xi(E,T)f(x,T) \end{array}\right.$$

A new integration constant A is introduced: $t_o = A/\Omega$. Then

$\Psi + A = (t + t_o)\omega(E,T)\xi = (t + t_o)\Omega$.

We introduce the new functions:

(10) $$\left|\begin{array}{l} x_c[E, \Psi + A, T] = x_d\left[E, \dfrac{\Psi + A}{\Omega}, T\right] \\ p_c[E, \Psi + A, T] = p_d\left[E, \dfrac{\Psi + A}{\Omega}, T\right] \end{array}\right.$$

From (4), (9) and (10) follows the system of equations:

$$\Omega\frac{\partial x_c(E, \Psi + A, T)}{\partial(\Psi + A)} - \xi(E,T)p_c = 0$$

$$\Omega\frac{\partial p_c(E, \Psi + A, T)}{\partial(\Psi + A)} + \xi(E,T)f(x_c,T) = 0$$

or, equivalently:

(11) $$\mathrm{Y}_{\mathrm{or}}\begin{bmatrix} 0 \\ \Omega \end{bmatrix} + \begin{bmatrix} -\xi \boldsymbol{p}_c \\ \xi \boldsymbol{f} \end{bmatrix} = 0$$

where:

$$(12)\qquad \Upsilon_{\text{or}}(\boldsymbol{E},\Psi+\boldsymbol{A},\boldsymbol{T})=\begin{bmatrix}\dfrac{\partial \boldsymbol{x}_c(\boldsymbol{E},\Psi+\boldsymbol{A},\boldsymbol{T})}{\partial \boldsymbol{E}} & \dfrac{\partial \boldsymbol{x}_c(\boldsymbol{E},\Psi+\boldsymbol{A},\boldsymbol{T})}{\partial(\Psi+\boldsymbol{A})}\\ \dfrac{\partial \boldsymbol{p}_c(\boldsymbol{E},\Psi+\boldsymbol{A},\boldsymbol{T})}{\partial \boldsymbol{E}} & \dfrac{\partial \boldsymbol{p}_c(\boldsymbol{E},\Psi+\boldsymbol{A},\boldsymbol{T})}{\partial(\Psi+\boldsymbol{A})}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial \boldsymbol{x}_c(\boldsymbol{E},\Psi+\boldsymbol{A},\boldsymbol{T})}{\partial \boldsymbol{E}} & \dfrac{\xi}{\Omega}\boldsymbol{p}_c\\ \dfrac{\partial \boldsymbol{p}_c(\boldsymbol{E},\Psi+\boldsymbol{A},\boldsymbol{T})}{\partial(\Psi+\boldsymbol{A})} & -\dfrac{\xi}{\Omega}\boldsymbol{f}(\boldsymbol{x}_c,\boldsymbol{T})\end{bmatrix}$$

and $\det \Upsilon_{\text{or}} = -\dfrac{\xi}{\Omega}$ (the condition for the applicability of the perturbation method is: $\Upsilon_{\text{or}} \neq 0,\infty$).

PERTURBING IN ENERGY-ANGLE VARIABLES

Below we solve the perturbed equation (1) applying the method of varying coefficients: we assume E = E(t) and A=A(t). We seek a solution of the form:

$$\left|\begin{array}{l} x = x_c[E(t),\Psi(t)+A(t),T] \\ p = p_c[E(t),\Psi(t)+A(t),T] \end{array}\right.$$

From here and (11) it follows:

$$\begin{bmatrix}\dfrac{dx}{dt}\\ \dfrac{dp}{dt}\end{bmatrix}=\Upsilon_{\text{or}}\begin{bmatrix}\dfrac{dE}{dt}\\ \Omega(t)+\dfrac{dA}{dt}\end{bmatrix}+\mu\begin{bmatrix}\dfrac{\partial x_c}{\partial T}\\ \dfrac{\partial p_c}{\partial T}\end{bmatrix}$$

Substituting in (1) and taking into account (11) gives:

$$(13)\qquad \begin{bmatrix}\dfrac{dE}{dt}\\ \dfrac{dA}{dt}\end{bmatrix}=\Upsilon_{\text{or}}^{-1}\begin{bmatrix}-\mu\dfrac{\partial x_c}{dT}\\ -\mu\dfrac{\partial p_c}{dT}+\mu F_v\end{bmatrix}-\begin{bmatrix}0\\ \dfrac{\xi-1}{\xi}\Omega\end{bmatrix},$$

where from (12) it follows:

$$\Upsilon_{\text{or}}^{-1}=\begin{bmatrix} f(x_c,T) & p_c\\ \dfrac{\Omega}{\xi}\dfrac{\partial p_c(E,\Psi+A,T)}{\partial E} & -\dfrac{\Omega}{\xi}\dfrac{\partial x_c(E,\Psi+A,T)}{\partial E}\end{bmatrix}$$

The system of equations (13) can be written in the form:

$$(14)\qquad \left|\begin{array}{l} \dfrac{dE}{dt}=\mu G_r(E,\Psi+A,t,T,\mu) \\ \dfrac{dA}{dt}=\mu G_s(E,\Psi+A,t,T,\mu)-\dfrac{\xi-1}{\xi}\Omega \end{array}\right.$$

We seek a solution having the asymptotic form (5) and (6) and also:

$$(15)\qquad \text{Å} = \text{Å}_{\hat{\imath}}(\grave{O})+\mu[E_1(T)+U_{r1}(t,T)]+\mu^2[E_2(T)+U_{r2}(t,T)]+\dots$$

$$(16)\qquad A = A_{\hat{\imath}}(\grave{O})+\mu[A_1(T)+U_{s1}(t,T)]+\mu^2[A_2(T)+U_{s2}(t,T)]+\dots$$

The condition is that U_{rk} and U_{sk}, k = 1,2,3,… should not contain secular terms, i.e.:

$$(17)\qquad \left\langle\frac{\partial}{\partial t}U_{rk}(t,T)\right\rangle_t = 0,\qquad \left\langle\frac{\partial}{\partial t}U_{sk}(t,T)\right\rangle_t = 0,\qquad k = 1,2,3,\dots,$$

where $\langle\ \rangle$ means average with respect to time t.

Substituting (15) and (16) in (14) we expand in series with respect to μ. In front of μ^1 we get:

$$\left|\begin{array}{l} \frac{dE_o(T)}{dT}+\frac{\partial U_{r1}(t,T)}{\partial t}=G_{r1} \\ \frac{dA_o(T)}{dT}+\frac{\partial U_{s1}(t,T)}{\partial t}=G_{s1}-\frac{1}{\mu}\frac{\xi-1}{\xi}\Omega \end{array}\right. .$$

On the right-hand side appear $E_o(T)$, $\Psi_o(T)$, $A_o(T)$, $\Omega_o(T)$, $\xi_o[E_o(T),T]$. Averaging with respect to time *t* and taking into account (17) gives:

$$\left|\begin{array}{l} \frac{dE_o}{dT}=\langle G_{1r}\rangle_t \\ \frac{dA_o}{dT}=\langle G_{1s}\rangle_t-\frac{1}{\mu}\frac{\xi_o-1}{\xi_o}\Omega \end{array}\right. \tag{18}$$

In addition to these equations expression (4) should also be considered. Then U_{r1} and U_{s1} can be found in analogy with the asynchronous case, described in [1]. Synchronization or frequency lock-on will be observed when condition $\frac{dA_o}{dT}=0$ is satisfied from which it follows that:

$$\langle G_s\rangle_t-\frac{1}{\mu}\frac{\xi_o-1}{\xi_o}\Omega_o=0 \tag{19}$$

From equations (4), (18) and (19) we find E_o, A_o, ξ.

Eventually, the initial approximation (18) is determined to have the form:

$$\begin{bmatrix} \left\langle \frac{dE}{dt}\right\rangle_t \\ \left\langle \frac{dA}{dt}\right\rangle_t \end{bmatrix}=\left\langle \mathrm{Y}_{\mathrm{or}}^{-1}\begin{bmatrix} -\mu\frac{\partial x_c}{dT} \\ -\mu\frac{\partial p_c}{dT}+\mu F_v \end{bmatrix}\right\rangle-\begin{bmatrix} 0 \\ \frac{\xi-1}{\xi}\Omega \end{bmatrix}, \tag{20}$$

where all the initial approximations have been substituted in.

PERTURBATION IN ACTION-ANGLE VARIABLES

We will first determine the generating system of Hamilton's canonical equations. The corresponding Hamiltonian has the form :

$$\hat{E}=K_a(x,p,T)=K_c[H(x,p,T),T]$$

where $K_c(H,T)=\int_0^H \xi(H',T)dH'$. The integral is taken at T = const. Here:

$$H=H(x,p,T)=\frac{1}{2}p^2+V(x,T)=E, \tag{21}$$

$\xi(H,T)$ is the corresponding detuning in the presence of synchronizing influence.

The canonical system of Hamilton's equations has the form:

$$\left|\begin{array}{l} \frac{dx}{dt}=\frac{\partial K_a(x,p,T)}{\partial p}=\frac{\partial K_c(H,T)}{\partial H}\frac{\partial H(x,p,T)}{\partial p}=\xi(E,T)p \\ \frac{dp}{dt}=-\frac{\partial K_a(x,p,T)}{\partial x}=\frac{\partial K_c(H,T)}{\partial H}\frac{\partial H(x,p,T)}{\partial x}=-\xi(E,T)f(x,T) \end{array}\right.$$

We look for a generating function $\boldsymbol{W_{or}(x,J,T)}$ such that the new momentum $\bar{p}$ is a constant, which we denote by J, i.e. J = $\bar{p}$. The new coordinate $\Psi=\bar{x}$ represents angle. Then:

$$p=\frac{\partial W_{or}(x,J,T)}{\partial x} \qquad \Psi=\frac{\partial W_{or}(x,J,T)}{\partial J} \tag{22}$$

The new Hamiltonian $\bar{k}$ coincides with the old Hamiltonian $\bar{k}(J,\Psi,T)=K(x,\rho,T)$. The period with respect to Ψ must be: $P_o=2\pi$. Then, from Hamilton's canonical equations it follows that:

$$\frac{\partial\Psi}{\partial t}=\frac{\partial\bar{K}(J,\Psi,T)}{\partial J}=const$$

$$\frac{\partial J}{\partial t}=-\frac{\partial\bar{K}}{\partial\Psi}=0\,,\quad J=\frac{1}{2\pi}\oint\frac{W_{or}}{\partial x}dx$$

The circular frequency in the generating system is also determined as $\frac{d\Psi}{dt}=\Omega_c(J,T)=\xi_c(J,T)\omega_c(J,T)$, where $\omega_c(J,T)$ is the circular frequency in the autonomous system at $\mu=0$, $\xi_c(J,T)=\xi[E(J,T),T]$. The oscillation period in the generating system is respectively: $\ddot{I}_{or}(J,T)=\frac{2\pi}{\Omega_c(J,T)}=\frac{\ddot{I}(E,T)}{\xi_c(J,T)}$, where $\Pi(E,T)$ is the autonomous system period.

Tanking into account (21) and (22) gives:

$$W_{or}(x,J,T)=\pm\int_0^x\sqrt{2E-2V(x',T)}dx'$$

$$\Psi=\int_0^x\frac{\omega_c}{\pm\sqrt{2E-2V(x',T)}}dx'=\Omega_c t+const$$

We now express x, p through the variables *J*, *Ψ*:

$$(23)\qquad\left|\begin{aligned}x&=x_f(J,\Psi+A,T)=x_c[E(J,T),\Psi+A,T]\\ p&=p_f(J,\Psi+A,T)=p_c[E(J,T),\Psi+A,T]\end{aligned}\right.$$

The generating system is then represented as:

$$\left|\begin{aligned}\frac{dx}{dt}&=\xi_c(J,T)p\\ \frac{dp}{dt}&=-\xi_c(J,T)f(x,T)\\ \frac{d\Psi}{dt}&=\Omega_c(J,T)\end{aligned}\right.$$

or in the following matrix form: $Z_{or}\begin{bmatrix}0\\ \Omega_o\end{bmatrix}+\begin{bmatrix}-\xi p_t\\ \xi f(x,T)\end{bmatrix}=0$,

where $Z_{or}(J,\Psi+A,T)=\begin{bmatrix}\frac{\partial x_f(J,\Psi+A,T)}{\partial J} & \frac{\partial x_f(J,\Psi+A,T)}{\partial(\Psi+A)}\\ \frac{\partial p_f(J,\Psi+A,T)}{\partial J} & \frac{\partial p_f(J,\Psi+A,T)}{\partial(\Psi+A)}\end{bmatrix}$

and det $Z_{or}=-1$ (the condition for applicability of the perturbation method in this case is $Z_{or}\neq 0,\infty$).

We now solve the perturbed system of equations (1). The perturbation is non-Hamiltonian. For this reason we approach it with the averaging method. We try to find a solution by varying the constants (23):

$$(24)\qquad\begin{bmatrix}\frac{dx}{dt}\\ \frac{dp}{dt}\end{bmatrix}=Z_{or}\begin{bmatrix}\frac{dJ}{dt}\\ \Omega_o(t)+\frac{dA}{dt}\end{bmatrix}+\mu\begin{bmatrix}\frac{\partial x_f}{\partial T}\\ \frac{\partial p_f}{\partial T}\end{bmatrix}$$

Substituting (24) in (1) we get:

$$\begin{bmatrix} \frac{dJ}{dt} \\ \frac{dA}{dt} \end{bmatrix} = Z_{or}^{-1} \begin{bmatrix} -\mu \frac{\partial x_f}{\partial T} \\ -\mu \frac{\partial p_f}{\partial T} + \mu F_v \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{\xi_c - 1}{\xi_c} \Omega_c \end{bmatrix} \tag{25}$$

where:

$$Z_{or}^{-1} = \begin{bmatrix} \frac{\xi_c}{\Omega_c} f(x_f, T) & \frac{\xi_c}{\Omega_c} p_f \\ \frac{\partial p_f}{\partial J} & -\frac{\partial x_f}{\partial J} \end{bmatrix}$$

The system of equations (25) can be written in the form:

$$\left| \begin{aligned} \frac{dJ}{dt} &= \mu G_V(J, \Psi + A, t, T, \mu) \\ \frac{dA}{dt} &= \mu G_W(J, \Psi + A, t, T, \mu) - \frac{\xi_c - 1}{\xi_c} \Omega_c \end{aligned} \right. \tag{26}$$

We now seek an asymptotic expansion:

$$\left| \begin{aligned} \Omega_c(T) &= \Omega_o(T) + \mu[\Omega_1(T) + W_1(t,T)] + \mu^2[\Omega_2(T) + W_2(t,T)] + \ldots \\ J(T) &= J_o(T) + \mu[J_1(T) + U_{V1}(t,T)] + \mu^2[J_2(T) + U_{V2}(t,T)] + \ldots \\ A(T) &= A_o(T) + \mu[A_1(T) + U_{W1}(t,T)] + \mu^2[A_2(T) + U_{W2}(t,T)] + \ldots \\ \xi_c(J,T) &= \xi_o(J,T) + O(\mu) \end{aligned} \right. , \tag{27}$$

where $U_{Vk}(t,T)$, $U_{Wk}(t,T)$, k = 1,2,3, … do not contain secular terms, i.e.

$\left\langle \frac{\partial}{\partial t} U_{Vk}(t,T) \right\rangle_t = 0,\quad \left\langle \frac{\partial}{\partial t} U_{Wk}(t,T) \right\rangle_t = 0$, k=1,2,3,…

We substitute (27) into (26) and expand in a series with respect to the powers of μ. We average with respect to time t. In front of μ^1 we get:

$$\left| \begin{aligned} \frac{dJ_o}{dT} &= \langle G_V \rangle_t \\ \frac{dA_o}{dT} &= \langle G_W \rangle_t - \frac{1}{\mu} \frac{\xi_o - 1}{\xi_o} \Omega \end{aligned} \right.$$

Then we find U_{V1} and U_{W1} etc.

We get frequency lock-on or synchronization when $\frac{dA_o}{dT} = 0$.

Finally, we obtain the following system of equations:

$$\left| \begin{aligned} \frac{dJ_o}{dT} &= \langle G_V \rangle_t \\ \langle G_W \rangle_t - \frac{1}{\mu} \frac{\xi_o - 1}{\xi_o} \Omega &= 0 \\ \frac{d\Psi}{dt} &= \Omega_c(T) = \xi \omega_c \end{aligned} \right. \tag{28}$$

Equations (28) determine J_o, A_o, ξ_o.

The initial approximation has the form:

(29) $$\begin{bmatrix} \left\langle \dfrac{dI}{dt} \right\rangle_t \\ \left\langle \dfrac{dA}{dt} \right\rangle_t \end{bmatrix} = \left\langle Z_{or}^{-1} \begin{bmatrix} -\mu \dfrac{\partial x_f}{dT} \\ -\mu \dfrac{\partial p_f}{dT} + \mu F_v \end{bmatrix} \right\rangle - \begin{bmatrix} 0 \\ \dfrac{\xi_c - 1}{\xi_c} \Omega_c \end{bmatrix}.$$

COMPARISON WITH KUZMAK'S METHOD IN MATRIX FORM

Above, in the consideration of a synchronous non-linear oscillation when using the non-canonical /non-Hamiltonian/ perturbation method in energy-angle variables as well as when using the canonical approach in action-angle variables, we assumed the detuning ξ in the system to be a known function. Here, on the contrary, ξ will initially be considered as an independent variable and any definite substitutions will only be made at a later stage.

We again consider a generating system in the form (9) where ξ=const is for now an independent parameter.

The solution of (9) is represented in the form:

(30) $$\begin{cases} x = x_a[E,(t+t_o)\xi,T] \\ p = p_a[E,(t+t_o)\xi,T] \end{cases}$$

We introduce an angle variable Ψ and the integration constant α according to the expressions: $\boldsymbol{t} = \dfrac{\Psi}{\omega(\boldsymbol{E},\boldsymbol{T})}$; $\boldsymbol{t_o} = \dfrac{\alpha}{\omega(\boldsymbol{E},\boldsymbol{T})}$, where $\omega(\boldsymbol{E},\boldsymbol{T}) = \dfrac{2\pi}{\Pi(\boldsymbol{E},\boldsymbol{T})}$ is the circular frequency in the generating solution, $\Pi(E,T)$ is the period in time t. Let $\theta = \Psi + \alpha$ and $\boldsymbol{N} = \begin{bmatrix} 1 & 0 \\ 0 & \xi \end{bmatrix}$..

We introduce the matrix $\Xi_g = \Xi[E,(t+t_o)\xi,T]$ such that it satisfies:

(31) $$\Xi_g \boldsymbol{N} = \begin{bmatrix} \dfrac{\partial x_a[E,(t+t_o)\xi,T]}{\partial E} & \dfrac{\partial x_a[E,(t+t_o)\xi,T]}{\partial (t+t_o)} \\ \dfrac{\partial p_a[E,(t+t_o)\xi,T]}{\partial E} & \dfrac{\partial p_a[E,(t+t_o)\xi,T]}{\partial (t+t_o)} \end{bmatrix}$$

We will work with the functions:

$$\begin{cases} x_b(E,\theta\xi,T) = x_a\left[E,\dfrac{\theta\xi}{\omega(E,T)},T\right] \\ p_b(E,\theta\xi,T) = p_a\left[E,\dfrac{\theta\xi}{\omega(E,T)},T\right] \end{cases}$$

Let:

$$\boldsymbol{Y}_g = \boldsymbol{Y}(E,\theta\xi,T) = \begin{bmatrix} \dfrac{\partial x_b[E,\theta\xi,T]}{\partial E} & \dfrac{\partial x_b[E,\theta\xi,T]}{\partial (\theta\xi)} \\ \dfrac{\partial p_b[E,\theta\xi,T]}{\partial E} & \dfrac{\partial p_b[E,\theta\xi,T]}{\partial (\theta\xi)} \end{bmatrix}$$

From (9), taking (30) into account, we obtain the following variational equation:

(32) $$\frac{\partial}{\partial t}(\Xi_g \boldsymbol{N}) + \xi \boldsymbol{B}(\Xi_g \boldsymbol{N}) = 0,$$

where $B = \begin{bmatrix} 0 & -1 \\ \dfrac{\partial f(x,T)}{\partial x} & 0 \end{bmatrix}$.

Apart from this, the quantity (31) can be represented also in the from:

(33) $$\boldsymbol{\Xi}_g \boldsymbol{N} = \boldsymbol{Y}_g \boldsymbol{H}\boldsymbol{N}[1-\xi(\boldsymbol{t}-\boldsymbol{t}_o)\boldsymbol{Q}],$$

where $H = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$, $\quad \boldsymbol{Q} = \begin{bmatrix} 0 & 0 \\ \dfrac{\partial \ln \Pi(\boldsymbol{E},\boldsymbol{T})}{\partial \boldsymbol{E}} & 0 \end{bmatrix}$

We now solve the system of equations (1). It is expressed in the following equivalent form:

(34) $$\left|\begin{aligned} &\frac{dx}{dt} - \xi p = (1-\xi)p \\ &\frac{dp}{dt} + \xi f(x,T) = \mu F_v + (\xi - 1)f(x,T) \end{aligned}\right.$$

We seek a solution in the form:

(35) $$\left|\begin{aligned} &x = x_b[E(T),\Psi(T) + A(T),T] + \mu U_{1a}(t,T) \\ &p = p_b[E(T),\Psi(T) + A(T),T] + \mu U_{2a}(t,T) \end{aligned}\right.$$

where $t_0 = \dfrac{A}{\Omega}$ and also:

(36) $$\frac{d\Psi(t)}{dt} = \Omega(t) \qquad \xi(E,t,T) = \frac{\Omega(t)}{\omega(E,T)}.$$

In this way we initially found Ξ and only then considered the constant period functions x_b, p_b.

Having /35/ as a basis to stand on we can write:

(37) $$\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dp}{dt} \end{bmatrix} = \mathrm{Y}_g \begin{bmatrix} \mu \dfrac{dE}{dT} \\ \Omega + \mu \dfrac{dA}{dT} \end{bmatrix} + \mu \begin{bmatrix} \dfrac{\partial x_b}{\partial T} \\ \dfrac{\partial p_b}{\partial T} \end{bmatrix} + \mu \frac{\partial \mathrm{U}_a}{\partial t} + \mu^2 \frac{\partial \mathrm{U}_a}{\partial T},$$

where we have introduced the notation: $\mathrm{U}_a(\boldsymbol{t},\boldsymbol{T}) = \begin{bmatrix} \mathrm{U}_{a1}(\boldsymbol{t},\boldsymbol{T}) \\ \mathrm{U}_{a2}(\boldsymbol{t},\boldsymbol{T}) \end{bmatrix}$

We look for a solution by representation with an asymptotic series:

$$\left|\begin{aligned} &\Psi(t) = \Psi_{»}(t) + \mu\Psi_1(t) + \mu^2\Psi_2(t) + \ldots \\ &\Omega_{љ}(t) = \Omega_{»}(T) + \mu[\Omega_1(T) + \omega_1(t,T)] + \mu^2[\Omega_2(T) + \omega_2(t,T)] + \ldots \\ &A(T) = A_{»}(t) + \mu A_1(T) + \mu^2 A_2(T) + \ldots \\ &E(T) = E_{»}(t) + \mu E_1(T) + \mu^2 E_2(T) + \ldots \\ &\xi(E,t,T) = \xi_o(E,T) + O(\mu) \\ &\mu\mathrm{U}_a(t,T) = \mu\mathrm{U}\ \ (t,T) + \mu^2\mathrm{U}_2(t,T) + \ldots \end{aligned}\right.$$

where $\mathrm{U}_k(\boldsymbol{t},\boldsymbol{T}) = \begin{bmatrix} \mathrm{U}_{k1}(\boldsymbol{t},\boldsymbol{T}) \\ \mathrm{U}_{k2}(\boldsymbol{t},\boldsymbol{T}) \end{bmatrix}$, k =1,2,3,…

From (9) and (35) it follows:

(38) $$\mathrm{Y_g}\begin{bmatrix}0\\ \Omega\end{bmatrix}+\begin{bmatrix}-\xi \boldsymbol{p_b}\\ \xi \boldsymbol{f}(\boldsymbol{x_b},\boldsymbol{T})\end{bmatrix}=0$$

Substituting /37/ in /34/ and taking into consideration /38/ we get:

$$\mu\left[\frac{\partial \mathrm{U}_{\boldsymbol{a}}}{\partial t}+\xi \mathrm{B}\mathrm{U}_{\boldsymbol{a}}\right]+\mu \mathrm{Y_g}\begin{bmatrix}\dfrac{\boldsymbol{dE}}{\boldsymbol{dT}}\\ \dfrac{\boldsymbol{dA}}{\boldsymbol{dT}}\end{bmatrix}+\mu\begin{bmatrix}\dfrac{\partial \boldsymbol{x_b}}{\partial \boldsymbol{T}}\\ \dfrac{\partial \boldsymbol{p_b}}{\partial \boldsymbol{T}}\end{bmatrix}=\mu\begin{bmatrix}0\\ \boldsymbol{F}_\nu\end{bmatrix}+(\xi-1)\begin{bmatrix}-\boldsymbol{p_b}\\ \boldsymbol{f}(\boldsymbol{x_b},\boldsymbol{T})\end{bmatrix}+\boldsymbol{O}(\mu^2)$$

Developing in a series in the powers of μ^K, k=1,2,3, … we get in front of μ^K:

(39) $$\frac{\partial \mathbf{U}_{\hat{e}}(t,T)}{\partial t}+\xi \mathbf{B}\,\mathbf{U}_{\hat{e}}(t,T)=\hat{\mathbf{O}}_{\hat{e}},\qquad \hat{e}=1,2,3,....$$

where $\Phi_{\mathrm{k}}=\begin{bmatrix}\Phi_{\mathrm{k1}}\\ \Phi_{\mathrm{k2}}\end{bmatrix}$ $\quad \boldsymbol{k}=1,2,3,...$.

In particular, in front of μ^1 we get:

(40) $$\frac{\partial \boldsymbol{U}_1(t,T)}{\partial t}+\xi \boldsymbol{B}\,\boldsymbol{U}_1(t,T)=\hat{\boldsymbol{O}}_1,$$

where

(41) $$\Phi_1=-\mathrm{У}_{\mathrm{g}}\begin{bmatrix}\dfrac{\boldsymbol{dE_o}(\boldsymbol{T})}{\boldsymbol{dT}}\\ \dfrac{\boldsymbol{dA_o}(\boldsymbol{T})}{\boldsymbol{dT}}\end{bmatrix}+\begin{bmatrix}-\dfrac{\partial \boldsymbol{x_b}}{\partial \boldsymbol{T}}\\ -\dfrac{\partial \boldsymbol{p_b}}{\partial \boldsymbol{T}}+\boldsymbol{F}_\nu\end{bmatrix}+\frac{(\xi_o-1)}{\mu}\begin{bmatrix}-\boldsymbol{p_b}\\ \boldsymbol{f}(\boldsymbol{x_b},\boldsymbol{T})\end{bmatrix}$$

The solution of (39) and, in particular of (40), is sought by variation of the constants. Let:

(42) $$\mathbf{U}_{\hat{e}}(t,T)=\Xi_g \mathbf{N}\,\mathbf{V}_{\hat{e}}$$

where $\mathrm{V_k}(\boldsymbol{t},\boldsymbol{T})=\begin{bmatrix}\mathrm{V_{k1}}(\boldsymbol{t},\boldsymbol{T})\\ \mathrm{V_{k2}}(\boldsymbol{t},\boldsymbol{T})\end{bmatrix}$.

Taking into account (32) from (39) it follows: $\dfrac{\partial \boldsymbol{V}_{\hat{e}}(t,T)}{\partial t}=(\Xi_g \boldsymbol{N})^{-1}\hat{\boldsymbol{O}}_{\hat{e}}$

and, taking into account (33),

(43) $$\boldsymbol{V}_{\hat{e}}(t,T)=\boldsymbol{V}_{\hat{e}}(0,T)+\int_0^t[1+(t'+t_o)\boldsymbol{Q}]\boldsymbol{N}^{-1}\boldsymbol{H}^{-1}\boldsymbol{Y}_g^{-1}\hat{\boldsymbol{O}}_{\hat{e}}\,dt'$$

Below, for brevity, the index “k” is omitted. From (43) it follows that:

(44) $$\mathrm{V}_{\hat{e}}(t,T)=\mathrm{V}_{\hat{e}}(0,T)+\int_0^t[1+(t'+t_o)\mathrm{Q}]\{\frac{\partial}{\partial t'}[\mathrm{K}_1+\mathrm{D}(T)t']\}dt'$$

where $\boldsymbol{K}_1$ and D(T) have been introduced through the relations:

$$\int_0^t \mathrm{N}^{-1}\mathrm{H}^{-1}\mathrm{Y}_{\boldsymbol{g}}^{-1}\Phi \boldsymbol{dt}=\mathrm{K}_1[\Psi(\boldsymbol{t}),\boldsymbol{T}]+\mathrm{D}(\mathrm{T})\boldsymbol{t},$$

$$\mathrm{D}(\mathrm{T})=\left\langle \mathbf{N}^{-1}\mathbf{H}^{-1}\mathbf{Y}_{\mathbf{g}}^{-1}\hat{\mathbf{O}}\right\rangle_t$$

at $\quad \mathbf{K}_1[\Psi(0),T]=0$

Integrating (44) by parts we obtain:

$$\mathbf{V}_k(t,T)=\mathbf{V}_k(0,\mathrm{T})+[1+(t+t_o)\mathbf{Q}]\{\mathbf{K}_1[\Psi(t),T]+\mathrm{D}(\mathrm{T})t\}-\mathbf{Q}\left\{\mathbf{K}_2[\Psi(t),\mathrm{T}]+\mathbf{L}(\mathrm{T})t+\frac{\mathrm{D}(\mathrm{T})t^2}{2}\right\}$$

where $\mathbf{L}(T)=\langle\mathbf{K}_1\rangle_t$.

$\int_0^t \mathbf{K}_1[\Psi(t'),T]dt'=\mathbf{K}_2[\Psi(t),T]+\mathbf{L}(T)t$ at $\mathbf{K}_2[\Psi(0),T]=0$.

Substituting in (42) and taking into account (33) as well as the fact that $\mathrm{Q}=0$, we get:

$$\mathbf{U}(t,T)=\mathbf{Y}_g\mathbf{H}\mathbf{N}\{-\frac{\mathbf{Q}\mathbf{D}t^2}{2}+[-\mathbf{Q}\mathbf{V}(0,T)+\mathbf{D}(T)-\mathbf{Q}\mathbf{L}(T)]t+[(1-t_o\mathbf{Q})\mathbf{V}(0,T)+\mathbf{K}_1-\mathbf{Q}\mathbf{K}_2]\}$$

The matrix function $\mathbf{U}$ will be periodic with respect to t under the condition that $\mathbf{QD}=0$ and $\mathbf{D}(T)=\mathbf{Q}[\mathbf{L}(T)]+\mathbf{V}(0,T)]$ for the satisfaction of which it is sufficient to do the substitution:

(45),(46) $\quad\mathbf{D}=0,\ \mathbf{V}(0,T)]=-\mathbf{L}(T)$.

Then we get:

$$\mathrm{U}(t,T)=\mathrm{Y}_g\mathrm{HN}\left[\mathrm{K}_1-\mathrm{QK}_2+(1-t_o\mathrm{Q})\mathrm{V}(0,T)\right]$$

The condition (45) is equivalent to:

(47) $$\left\langle \mathbf{Y}_g^{-1}\hat{\mathbf{O}}_{\hat{e}}\right\rangle_t=0,\qquad \hat{e}=1,2,3,\ldots$$

and, in particular, to the first order (μ^1). When k=1, from (41) and (47) it follows that:

(48) $$\begin{bmatrix}\left\langle\dfrac{d\mathring{A}_{\hat{\imath}}(\grave{O})}{d\grave{O}}\right\rangle_t\\ \left\langle\dfrac{dA_{\hat{\imath}}(\grave{O})}{d\grave{O}}\right\rangle_t\end{bmatrix}=\left\langle\mathbf{Y}_g^{-1}\begin{bmatrix}-\dfrac{\partial x_b}{dT}\\ -\dfrac{\partial p_b}{dT}+F_\nu\end{bmatrix}\right\rangle_t-\begin{bmatrix}0\\ \dfrac{\Omega_c(\xi_c-1)}{\mu\xi_c}\end{bmatrix}.$$

In this way we eventually obtained the system of equations (36), (46) and (47).

CONCLUSION

The analysis of the results obtained above leads to the important conclusion that: first, the initial approximation (29) obtained in action-angle variables is equivalent to the initial approximation (20) obtained in energy-angle variables; and, second, the equations obtained by Kuzmak's method are equivalent in first approximation /μ^1/ to the corresponding equations obtained by the non-canonical perturbation approach in energy-angle variables /see equations (48) and (20)/. The obtained results support the idea, particularly in the context of the analysis of a non-linear oscillator under external synchronizing influence, that the non-canonical /non-Hamiltonian/ and the canonical /Hamiltonian/ methods do not differ in principle. We must note at this point that, in the theory of nonlinear oscillations, a number of other methods exist that are not, even in first approximation, absolutely equivalent to the three methods presented above.

REFERENCES

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NON-LINEAR OSCILLATOR UNDER EXTERNAL SYNCHRONIZING INFLUENCE: COMPARISON OF PERTURBATION AND CANONICAL METHODS OF ANALYSIS

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Abstract. The term “Synchronous action” here encompasses all cases of frequency lock-on, frequency multiplication and division, phenomena of synchronization at commensurable frequencies. A non-canonical /non-Hamiltonian/ perturbation approach is presented for the study of non-linear oscillator under external synchronous action in “energy-angle” variables. The iteration constants of the initial solution are introduced to be the new variables. By applying consistently the method of canonical transformations and multiplying functions, a canonical approach in “action-angle” variables for analysis of the same system under similar conditions is developed. Both approaches are characterized by the transition, in the very beginning, to functions with constant period, and only then the necessary functional matrices are introduced. The same problem is studied on the basis of a version of Kuzmak’s method developed in a matrix form, for the case when, in the very beginning of the study, the system detuning is regarded to be an independent parameter. The conclusion is formed, on the basis of the performed analysis, that the equations and the first approximation of the respective solutions when using the three basic approaches mentioned above are equal to each other. In particular, this conclusion is a contribution to the idea that there is no principal difference between the non-canonical /non-Hamiltonian/ and the canonical /Hamiltonian/ methods. However, attention is drawn to the fact that some of the other existing analytical methods developed in the frames of the Theory of Non-linear Oscillations could not give, even in the first approximation, a complete coincidence with the solution obtained using the three approaches mentioned above.