Българска академия на науките. Bulgarian Academy of Sciences
Аерокосмически изследвания в България. 15. Aerospace Research in Bulgaria
София. 1999. Sofia

Qualitative analysis of the free processes in a generalized linear oscillating circuit with periodic parameters. Part 3. Analysis of the free processes in piece-wise linear and quasi-harmonic oscillating circuit [1]

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Introduction

The general analysis of the stationary modes of oscillating systems with periodic parameters is of great theoretical and practical significance. Similar obstacles crop up in connection with many problems related to the Theory of oscillations in particular when investigating parametric amplification and generation of oscillations, frequency modulation, detection and conversion, suppression of undesired oscillations and intermodulation distortions, etc.

In the Part 1of the paper [1], the possible transformations of the equations of an oscillating circuit with periodic and almost periodic parameters have been given and the expedience of using different equations forms has been analyzed. A qualitative picture of the free processes in an oscillating circuit has been presented on the basis of the mathematical theory of Hamiltonian systems.

The Part 2 of the paper [2] has been focused on the problems of the stability of the cannonical systems in a general form. Criteria for the stability or instability of a general linear resonance circuit have been formed.

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[1] An investigation, supported by the "Scientific Research" Bulgarian National Fund under Contract No TH - 549/95.

Present Part 3 of the paper provides an analysis of the free processes, in piece-wise linear and quasi-harmonic oscillating circuits, from the point of view of their bounded or unbounded nature. In other words, it studies issue of the stability and instability of the oscillating circuit.

Stability of an oscillating circuit with a piece-wise linear volt-coulomb characteristic

In linear approach the concept of stability coincides with the concept boundedness of all solutions of the respective differential equation with a zero right-hand part and bounded initial conditions. Having lost its stability, the parametric amplifier is converted into a parametric oscillator.

The equation of a series oscillating circuit containing capacitance $C(t_r)$, periodically changing in time, acquires the following form with respect to the charge q

$$\frac{d^2q}{dt_r^2}+\frac{R}{L}\frac{dq}{dt_r}+\frac{1}{LC(t_r)}q=0\,, \tag{1}$$

where R and L are the constant resistance and the inductance of the oscillating circuits.

We use the following substitution $q_1 = q\exp(-\frac{1}{2}\frac{R}{L}t_r)$ to reduce Eq. (1) to the form:

$$\frac{d^2q_1}{dt_r^2}+\left[\frac{1}{LC(t_r)}-\frac{1}{4}\frac{R^2}{L^2}\right]q_1=0\,, \tag{2}$$

which contains no explicit dissipative term.

The capacitance $C(t_r)$ is regarded as a piece-wise linear time function,

$$C = C_1 \text{ for } t_r \in (0,t_{r1})\text{, and } C = C_2 \text{ for } t_r \in (t_{r1},T_r)\,,$$

where T_r is the capacitance change period. Thus the capacitance changes twice in one period: when $t_r = t_{r1}$, it leaps from C_1 to C_2, and when $t_r = T_r$, from C_2 to C_1.

Equation (2) can be written in the following dimensionless form:

$$\frac{d^2y}{dt^2}+a(t)y=0\,, \tag{3}$$

where

$$t=\frac{t_r}{t_{oo}},\quad t_{oo}=\sqrt{L}\,\sqrt[4]{C_1C_2}\,,\quad a(t)=\alpha^2=\sqrt{\frac{C_2}{C_1}}$$

at $t\in(0,t_1)$, and $a(t)=\frac{1}{\alpha^2}=\sqrt{\frac{C_1}{C_2}}$ at $t\in(t_1,T)$, $t_1=\frac{t_{r1}}{t_{oo}}$, $T=\frac{T_r}{t_{oo}}$.

Equation (3) has constant coefficients in the interval $(0,t_1)$ and its solution is a cosine curve. The same can be established for the interval (t_1,T). At moment t_1 the amplitude, the frequency and the initial phase of the cosinusoid change in a leap-like way, yet so that the function turns out to be continuous and sufficiently smooth (with continuous first-order derivative). Equation (3) is a particular case of Hill's equation. It follows from the latter's theory that it cannot be asymptotically stable, i.e. if $y(t)$ is a solution of the equation, $\lim_{t\to\infty} y(t) \neq 0$.

Therefore, the stability problem boils down to specifying one of the two cases: a) the case of stability, when the above-mentioned boundary is a finite number; b) the case of instability, when the module of the boundary is equal to infinity.

The solution of Eq. (3) is determined in the following way:

$$
\begin{aligned}
y(t) &= y_{01}\cos(\alpha t+\varphi_1) \quad \text{for} \quad t\in(0,t_1),\\
y(t) &= y_{02}\cos(\frac{1}{\alpha}t+\varphi_2) \quad \text{for} \quad t\in(t_1,T),
\end{aligned} \tag{4}
$$

y_{01} and φ_1 are determined by the initial conditions, y_{02} and φ_2 are specified by the condition of continuousness of the function $y(t)$ and its derivative at the moment t_1, when the capacitance changes in a leap-like manner.

$$\varphi_2 = \operatorname{arctg}[\alpha^2 \operatorname{tg}(\alpha t_1+\varphi_1) - \frac{1}{\alpha}t_1],$$

$$y_{02} = \alpha^2 y_{01}\frac{\sin(\alpha t_1+\varphi_1)}{\sin(\frac{1}{\alpha}t_2+\varphi_2)}.$$

Equation (3) can be presented in a matrix form:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad y_1 = y, \quad y_2 = \frac{dy}{dt}$$

or in short :

$$\frac{d}{dt}\mathbf{Y} = \mathbf{A}(t)\mathbf{Y}. \tag{5}$$

According to Flocker's theory, the fundamental matrix of the solution satisfies the condition:

$$\mathbf{Y}(t+T) = \mathbf{Y}(t)\mathbf{Y}(T), \tag{6}$$

where $\mathbf{Y}(T)$ is a constant matrix, called monodromy matrix. This matrix allows of obtaining Lyapounov's constant $a = \mathrm{Sp}\mathbf{Y}(T)$, where Sp is the sum of the elements of the principal diagonal of the matrix.

In accordance with Lyapounov's first method, the following three cases are distinguished for Eq. (3): 1) $|a| > 2$, 2) $|a| < 2$, 3) $|a| = 2$.

In the first case the equation is unstable, in the second one it is stable, and in the third case it is determined by the boundary between the stable and unstable ranges.

Using solution (5), which can serve for construction the monodromy matrix, whose elements are obtained in the form:

$$y_{11} = \cos\alpha t_1 \cos\frac{1}{\alpha}(T - t_1) - \alpha^2 \sin\alpha t_1 \sin\frac{1}{\alpha}(T - t_1),$$

$$y_{12} = \frac{1}{\alpha}\sin\alpha t_1 \cos\frac{1}{\alpha}(T - t_1) + \alpha\cos\alpha t_1 \sin\frac{1}{\alpha}(T - t_1),$$

$$y_{21} = -\alpha\cos\alpha t_1 \sin\frac{1}{\alpha}(T - t_1) - \frac{1}{\alpha}\sin\alpha t_1 \cos\frac{1}{\alpha}(T - t_1),$$

$$y_{22} = -\frac{1}{\alpha^2}\sin\alpha t_1 \sin\frac{1}{\alpha}(T - t_1) + \cos\alpha t_1 \cos\frac{1}{\alpha}(T - t_1).$$

This helps derive an expression for Lyapounov's constant :

$$a = \mathrm{Sp}\,\mathrm{Y}(T) = y_{11} + y_{22} = 2\cos\alpha t_1 \cos\frac{1}{\alpha}(T - t_1) - (\alpha^2 + \frac{1}{\alpha^2})\sin\alpha t_1 \sin\frac{1}{\alpha}(T - t_1)\,. \tag{7}$$

This expression allows of identifying which of the above-listed three cases the system refers to in each specific occurrence and in what way the stability problem is to be solved. The areas of stability or instability are determined by three parameters — α, t_1 and T, i.e. in the general case they can be plotted in a three-dimensional space.

Fig.1 shows the areas of stability and instability (hatched) on the plane $\left(\alpha, \frac{t_1}{2\pi}\right)$ for the particular case of $T = 2\pi$. Parameter α changes within the range $1 \le \alpha \le 10$. It can be obtained from the case for $\alpha > 1$, given above, by carrying out the following substitution of the parameters: $\beta = \frac{1}{\alpha}, \tau_1 = T - t_1$.

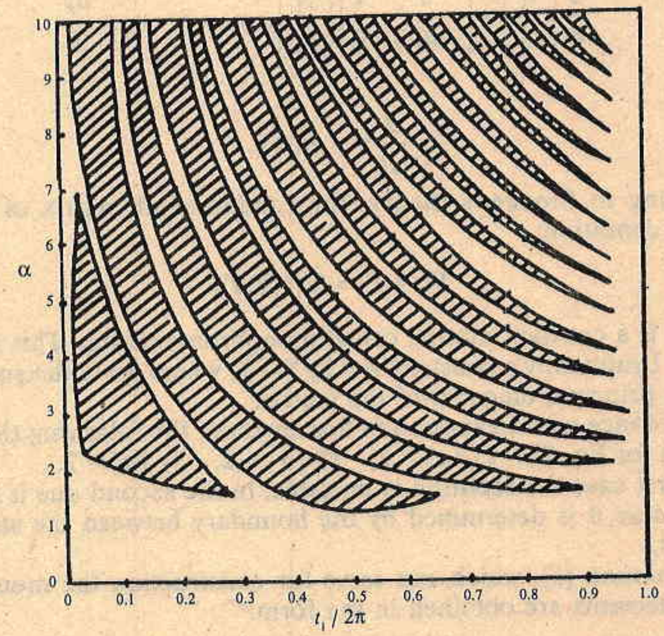

Fig.1

Analysis of the free processes in a linear quasi-harmonic oscillating circuit

Object of the analysis in this section is a linear quasi-harmonic oscillating circuit, whose free processes are described by a quasi-harmonic time function, i.e. by a sine function with amplitude slowly changing in time and first derivative of the phase by time. The major application area of such oscillating circuits is that of harmonic signal modulators.

Let us consider the general case of a series resonance circuit, where the inductance L, capacitance C and resistance R are time-dependent. The free process in such a circuit is described by the equation

$$\frac{d^2x}{dt^2}+2\alpha(t)\frac{dx}{dt}+\omega_o^2(t)x=0, \tag{8}$$

where $x=\frac{q}{q_{oo}}, t=\frac{t_r}{t_{oo}}$ — normalized capacitor charge and normalized time respectively, $\alpha=\frac{dL}{dt}+t_{oo}R$, $\omega_o(t)=\frac{t_{oo}}{\sqrt{LC}}$.

The general solution of Eq. (8) can be presented in the form

$$x=A(t,\tau)\sin\varphi(t,\tau), \tag{9}$$

where

$$A(t,\tau)=\frac{1}{\sqrt{\omega_o(t)}}\frac{1}{\sqrt{\omega_o(\tau)}}e^{-\beta(t,\tau)},$$

$$\beta(t,\tau)=\int_\tau^t[\alpha+(\alpha+\frac{1}{2}\frac{d}{dt}\ln\omega_o)\cos 2\varphi(t,\tau)]dt, \tag{10}$$

$$\varphi(t,\tau)=\int_\tau^t[\omega_o+(\alpha+\frac{1}{2}\frac{d}{dt}\ln\omega_o)\sin 2\varphi(t,\tau)]dt\cdot$$

Here the constant τ determines an arbitrary initial point of time.

Equation (10) given above yields an indicator of the integrability of Eq. (8) in quadratures —

$$\alpha(t)+\frac{d}{dt}\ln\omega_o(t)=0.$$

The first and second equations of (10) can be produce an indicator of the asymptotic stability of the oscillating circuit — $\frac{d\omega_o(t)}{dt}<0$ for $t>t'$, t' — an arbitrary moment.

So far we did not impose any limitations on the laws of time-dependent oscillating circuit parameters.

Let us suppose that the oscillating circuit satisfies the quasi-harmonic condition, i.e. we assume that the first co-multiplier in the right-hand part of (9) is a slowly changing time-dependent function. Under these conditions the approximate extremums of the solution are

$$\varphi(t_k,\tau) = (2k+1)\frac{\pi}{2},\ k = 0, \pm 1, \pm 2, \ldots,$$

respectively the zeroes of the solution are $\varphi(t_k^o,\tau) = k\pi$.

The logarithmic damping decrement is determined by using two adjacent extremums of the solution

$$\nu(t_k) = \ln\frac{A(t_{k-1},\tau)}{A(t_k,\tau)} = \int_{t_{k-1}}^{t_k} (\alpha + \frac{1}{2}\frac{d}{dt}\ln\omega_o)\cos^2\varphi dt.$$

The non-negativity of the logarithmic damping decrement for any t_k is an evidence of stability, and its positiveness is an evidence of asymptotic stability of the quasi-harmonic oscillating circuit.

The last expression can yield an analytical indicator of stability of the quasi-harmonic oscillating circuit:

(11) $$\alpha = \frac{1}{2}\frac{d}{dt}\ln\omega_o \geq 0.$$

If we introduce the parameter instantaneous characteristic resistance of the oscillating circuit $\rho = \sqrt{\frac{L}{C}}$, (11) can be used to obtain a simple criterion of asymptotic stability — $\rho = \text{const}$.

We introduce the concept instantaneous quality factor of the oscillating circuit $Q = \frac{\rho}{R}$, and as a result the formula of the logarithmic damping decrement acquires the form

$$\nu(t_k) = \int_{k\pi-\frac{\pi}{2}}^{k\pi+\frac{\pi}{2}} \frac{\cos^2\varphi}{Q+0{,}5\sin 2\varphi}(d\varphi + Qd\ln\rho).$$

Let $Q = \text{const}$. Then

$$\nu(t_k) = \int_{k\pi-\frac{\pi}{2}}^{k\pi+\frac{\pi}{2}} \frac{\cos^2\varphi d\varphi}{Q+0{,}5\sin 2\varphi} + \int_{\ln\rho(t_{k-1})}^{\ln\rho(t_k)} \frac{Q\cos^2\varphi}{Q+0{,}5\sin 2\varphi} d(\ln\rho).$$

The first integral is equal to $\pi(4Q^2-1)^{-1/2}$. Hence, the stability criterion can be written as follows:

$$\int_{\ln\rho(t_{k-1})}^{\ln\rho(t_k)} \frac{Q\cos^2\varphi}{Q+0{,}5\sin 2\varphi} d(\ln\rho) < \frac{\pi}{\sqrt{4Q^2-1}}.$$

Since $\max\limits_{\varphi} \frac{Q\cos^2\varphi}{2Q+\sin 2\varphi} = \frac{2Q^2}{4Q^2-1}$, after substituting the sub-integral expression for its maximum value, we obtain

$$\frac{4Q^2}{4Q^2-1}\ln\frac{\rho(t_k)}{\rho(t_{k-1})} < \frac{\pi}{\sqrt{4Q^2-1}}.$$

This inequality allows of obtaining the following simple stability expression at $Q = \text{const}$.

$$\ln\frac{\rho_{\max}}{\rho_{\min}} < \frac{\pi}{2Q}.$$

For a more detailed investigation of the free processes in an oscillating circuit with slowly changing parameters it is desirable to obtain an approximate solution of the last integral equation in (10). It is presented in the following way:

$$\varphi = \gamma + \varepsilon, \quad \gamma = \int_{\tau}^{t} \omega_o dt, \quad \varepsilon = \int_{\tau}^{t} (\alpha + \frac{1}{2}\frac{d}{dt}\ln\omega_o)\sin 2\varphi dt.$$

The inequality $\gamma >> \varepsilon$ is valid for many cases of practical importance, since the sub-integral function of the first integral has a constant sign, while that of the second one is quickly oscillating. After differentiating the last equation in (10), we obtain

$$\frac{d\varepsilon}{dt} = (\alpha + \frac{1}{2}\frac{d}{dt}\ln\omega_o)(\sin 2\gamma\cos 2\varepsilon\alpha\cos 2\gamma\sin 2\varepsilon).$$

In linear approximation $(\cos 2\varepsilon \cong 1, \sin 2\varepsilon \cong 2\varepsilon)$ this equation is solved in quadratures

$$\varepsilon = \left[2\int_{\tau}^{t}\mu\cos 2\gamma dt\right]\int_{\tau}^{t}\mu\sin 2\gamma\exp\left[-2\int_{\tau}^{t}\mu\cos 2\gamma\right]dt.$$

After obtaining ε equations (10) can be computed in quadratures and in this way the problem is completely solved.

Conclusion

The phenomenon "resonance" occupies peculiar position in natural and other sciences, in technology, in civil engineering subjects, in medicine, in the theory of musical instruments, in aeronautics theory, in rocket technics and astronautics, etc. Resonance is often manifested in the world that surrounds us either as a highly useful phenomenon or as an extremely harmful one. Radio communications, radio broadcasting, television and the other radio engineering systems would be absolutely inconseivable without resonance. Resonance is quite multiaspectual and multiform even in oscillating circuits with constant parameters. Resonance phenomena in nonlinear oscillating systems are virtually boundless.

Linear systems are quite frequently identified as systems with constant parameters not only in textbooks and other teaching aids but in scientific works as well. This approach reduced drastically the class of linear systems since it excludes linear systems with time-dependent parameters. The principle of linear connection, formulated relatively recently, has boosted the significance of linear systems with variable parameters, since it follows from this principle that if the whole set of linear systems can be studied, this will automatically lead to the establishment of the necessary scientific basis for investigating the processes occurring in nonlinear systems.

Oscillating circuits with periodic parameters can be divided into two groups. One of them includes oscillating circuits where given arbitrary initial conditions, free processes are limited. The other group, respectively, encompasses oscillating circuits, whose initial conditions can be selected in such a way as to ensure unlimited free processes. Each group of oscillating circuits, in its turn, is characterized by a set of stability and instability area. It is particularly important to develop analytical approaches for determining the area of stability or instability to which the specific oscillating circuit belongs.

The paper contains formulations of general theorems on systems with positive elements concerning the relation between the parameters of the system and the matrix elements of the respective differential vector equation. Infinitive systems of algebraic equations for some typical systems with periodic parameters are obtained; the properties of these systems of equations are explored, the general case relation between the complication of the radio-physical systems and the respective alteration of the systems of equations describing them is identified. The multitude of free processes occurring in an oscillating circuit with periodic and almost periodic parameters is visualized in a dynamic picture. Theorems concerning the conditions that would be sufficient for the stability or instability of an oscillating circuit with periodic parameters are formulated.

Qualitative analysis assumes considerable importance in the investigation of complex oscillating systems, since it allows of identifying the most general features of systems behaviour. Such a general perspective of the approach makes it interesting from a practical point of view, since the oscillating circuit with periodical parameters is quite rich in particular cases.

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Received 6.XI.1996

Качествен анализ на свободните процеси в обобщен линеен трептящ кръг с периодични параметри. Част 3. Анализ на свободните процеси в интервално-линейна и квази-хармонична трептяща верига

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(Резюме)

В част първа и в част втора на статията бе доказана необходимостта от изследване на трептящите резонансни системи с периодични и почти периодични параметри в общ вид с оглед цялостно обхващане на огромното разнообразие от режими, закономерности и свойства. Разработен бе общ метод за анализ на такива системи, които бяха групирани и класифицирани в тримерно цилиндрично пространство на параметрите по техния най-важен признак — устойчивостта или неустойчивостта по Ляпунов. Проблемите на устойчивостта бяха изследвани както в най-общ вид с използване на канонична система уравнения, така и при определена конкретизация на трептящата резонансна система с периодични параметри.

Трета част на работата продължава излагането на методите за използване разработения аналитичен подход към достатъчно конкретизирани резонансни системи с периодични и почти периодични параметри. Изследват се свободните трептящи процеси в интервално-линейна резонансна система, в която конкретно капацитетът приема две определени стойности със зададен период във времето. Друга конкретизирана задача е изследване на свободните процеси в линейна квазихармонична трептяща система, в която движението се описва с квазихармонична функция (използвана е синусна функция с бавноизменяща се амплитуда). И в двата случая са формулирани общи теореми относно съотношението на положителните параметри на системата и елементите на матрицата на съответното векторно диференциално уравнение. Изведени са условията за устойчивост и неустойчивост на разглежданите системи.