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Elliptical accretion discs with constant eccentricity. II. Standard α-disc model

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I. Introduction

In 1970's Shakura and Sunyaev [1] have developed a stationary accretion disc model widely known as a "standard α-disc". They have introduced a parameter α which simultaneously describes the efficiency of two of the most important angular momentum transport mechanisms. Namely, the turbulent viscosity and the magnetic field viscosity. The parameter α gives the proportionality relation between the shear stress $w_{r\varphi}$ and ρv_s^2, where ρ is the gas density and v_s is the sound speed. Values of α under an astrophysical interest lie in the range $10^{-5} \leq \alpha \leq 1$. It should be noted that during the type I and type II X-ray burst events the intensive flow of low angular momentum X-ray photons from the neutron star surface is able to increase strongly mass accretion rate $\dot{M}$, although the radiation pressure is acting to push out the infalling matter [2—4]. Such an obviously nonstationary mechanism of angular momentum release from the inner disc matter is not included in the standard α-disc model, where the radiative and molecular viscosities are neglected [1]. The importance of radiative viscosity is also mentioned by Arav and Begelman [5], but these cases include considerations of very hot innermost disc regions and jets arising from supercritical accretion regime. Pringle [6] has recently discussed the present status of the angular momentum transport problem in accretion discs. It should be expected that changing of the physical conditions naturally leads to different values of α. For example, the radially moving thermal instability front can delimit the accretion disc into relatively "cool" region with $\alpha=\alpha_c$ and "hot" part with $\alpha = \alpha_h$ [7, 8]. We shall not consider in this paper such a possible type of instability and shall neglect the radial dependence of α at all. We shall also neglect the vertical disc structure (the z-coordinate dependence), using disc-height averaged quantities (temperature T, densiry ρ, etc.).

The standard α-model is described by Shakura and Sunyaev [1] for steady-state circular discs. Recently Syer and Clarke have derived solutions for stationary discs with constant eccentricities in the form of the two-dimensional analogues of the α-model ([9], Appendix B). This generalization of the standard α-disc model was related with the observational and theoretical evidences that the tidal interaction in some close binary stellar systems may cause elongation of the accretion discs (mostly of their outer parts) [10—12].

II. Accretion disc model

Lyubarskij et al. [13] have investigated analitically and numerically viscous accretion discs consisting of series of nested elliptical streamlines. For every stremline with a major semiaxes a, eccentricity e and focal parameter $p=a(1-e^2)$, the viscosity coefficient η, temperature T, pressure P and volume density ρ were averaged over the disc thickness H. Lyubarskij et al. [13] confirmed Syer and Clarke's [9,14] conclusion about the longevity of eccentric discs in a Keplerian gravitational potential also in the case of variable eccentricity $e=e(p)$. It should be noted that these inferences are not influenced by the initial conditions which describe the accretion disc bearing. Throughout this paper we shall investigate the problem of the accretion disc radial structure only in the particular case of constant eccentricity e (i.e., $\partial e/\partial p \equiv e_p = 0$).

Following Lyubarskij et al. [13], instead of the usual polar coordinates (r, φ), we shall use the nonorthogonal curvilinear Eulerian coordinates (p, φ). The transformation between them can easily be performed by means of the relation

$$p = r\,(1+e\cos\varphi), \tag{1}$$

where φ is measured with respect to the direction of pericentre, which is assumed to be the same for all particle streamlines. According to above expression, if any quantity (e.g., disc surface density Σ) is a constant around the streamline, but depends on the focal parameter p, a transition to the radial coordinate r (i.e., the distance from the centre of the compact object) recovers the angular dependence on the azimuthal angle φ. The considered model uses the approximation of "pressure-free" fluid in the sense that the pressure does not play direct role in the dynamics of the flow, except that it controls the local viscosity. This allows a quasi-Keplerian motion of the gas particles, which spiral inward with a low radial drift velocity v^p. The later is connected to the local accretion rate $\dot{M}$ ([13], eq. (30)), assumed to be a constant with respect to p and time t, as we deal *ab initio* with the stationary accretion problem. Our consideration includes the assumption that both H and T are in local equilibrium (case C from the model of Syer and Clarke [9, 14]), which means that these quantities change fast enough along the streamlines in order to attain a correspondence with the other local physical conditions. On the other hand, neglecting the radial heat transfer, we shall assume that the heat losses are slow. This enables us to accept an adiabatic motion of the gas during its nearly Keplerian orbiting along the streamlines:

(2) $$P(p, \varphi) = K(p, \varphi)\, \rho^{\gamma}(p, \varphi),$$

where K is the specific gas entropy. The power γ is the adiabatic index and assumes values 5/3, 7/5 and 4/3 for an ideal gas consisting of one, two or more than two kinds of particles, correspondingly. By analogy with the standard α-disc model [1], the eccentric accretion disc can also be divided into three different zones [13]:

(i) innermost zone A with a radiation dominated plasma ($\gamma = 4/3$) and opacity k determined by Thomson scattering

(3) $$k = k_T = \sigma_T / m_p,$$

where σ_T is the Thomson cross-section and m_p is the proton mass. The equation of state is given by

(4) $$P = \frac{4\sigma_B}{3c} T^4,$$

where c is the light speed and σ_B is the Stefan-Boltzmann constant;

(ii) middle zone B with an ideal gas dominated plasma with equation of state

(5) $$P = \frac{R}{\mu} \rho T,$$

where R is the universal gas constant and μ is the mean molecular weight. If the plasma is fully ionized, then $\mu = 1/2$ (pure hydrogen composition), $\mu = 4/3$ (pure helium composition), and $\mu \approx 2$ if only heavier elements present. For cosmic abundance $\mu \approx 0.6$. For the adiabatic index we assume value $\gamma = 5/3$. The opacity law for this zone is Thomson scattering;

(iii) outermost zone C with an ideal gas dominated plasma (equation of state (5)) and free-free absorption

(6) $$k = k_{ff} = \zeta \rho T^{-7/2} \approx 6.273 \times 10^{22} \rho T^{-7/2},$$

where ζ is a numerical constant, which value depends on the mean Gaunt factor ($\bar{g} \approx 1$) and the chemical composition [15]. The adiabatic index is accepted to be equal to $\gamma = 5/3$.

Everywhere in this paper we shall use lower indices A, B and C to label quantities which are referred to one of the above mentioned zones, respectively. We shall also describe briefly the following four equations, giving an additional information which is enough to specify the structure of the constant eccentricity elliptical α-disc model. More details about these relations may be found in ([9], Appendix B) and in [13]. Note that the form of these expressions is independent of what of the three disc zones is under investigation, except the values of the integration constants $D_A(e)$, $D_B(e)$ and $D_C(e)$ which must be evaluated from the corresponding boundary conditions.

The equation for angular momentum balance may be written as

(7) $$\int_0^{2\pi} \eta(p, \varphi, e)\, Y(e, \varphi)\, d\varphi = \frac{2}{3}\left(\dot{M} - \frac{D(e)}{\sqrt{GMp}} \right),$$

where $D(e) = D_A(e), D_B(e)$ or $D_C(e)$; G is the gravitational constant, M is the mass of the compact object in the disc centre and

(8) $$Y(e,\varphi)=\frac{1}{3\,(1+e\cos\varphi)^3}\left[(3+e^2)+(7e+e^3)\cos\varphi+4e^2\cos^2\varphi\right],$$

(for $e_p = 0$) is an auxiliary function [13, 16]. Taking into account that the vertically averaged volume density is $\rho=\Sigma/H$ and the Keplerian angular frequency is $\omega_K=\sqrt{GM/r^3}=\sqrt{GM/p^3}\,(1+e\cos\varphi)^{3/2}$, we can express the condition of hydrostatic equilibrium in the z-direction of the disc as

(9) $$\frac{P(z=0)}{\rho H}=\frac{P(z=0)}{\Sigma}=\frac{1}{8}\,\omega_k^2 H.$$

In the third place, the heat removal by radiation from the accretion disc is described by the thermal balance equation. According to Shakura and Sunyaev ([1], eq. (2.7)), if Q is the energy flux radiated from unit disc surface per unit time, then it is related to the energy density of radiation ε_r by

(10) $$\frac{3}{4}\frac{Q}{c}k\Sigma=\varepsilon_r\equiv\frac{4\sigma_B}{c}\,T^4.$$

Therefore, the radiation losses from the two boundaries (upper and lower) of the disc contour per unit time are

(11) $$\int_0^{2\pi}\frac{32\sigma_B T^4}{3k\rho\,H}\sqrt{g}\,d\varphi\,dp.$$

In accordance with our adiabatic approach, we shall neglect the work done over the contour by the outer forces and shall retain only the term describing the heat generation inside the contour by the viscous forces ([13], eq.(16)). Consequently, supposing the validity of the diffusion approximation, the thermal balance equation becomes

(12) $$\frac{1}{2}\int_0^{2\pi}\eta\sigma_{ik}\sigma^{ik}\sqrt{g}\,d\varphi=\int_0^{2\pi}\frac{32\sigma_B T^4}{3k\rho H}\sqrt{g}\,d\varphi,$$

where σ^{ik} is the shear tensor, related to the viscous stress tensor w^{ik} by means of the viscosity coefficient η ($w^{ik}=\eta\sigma^{ik}; i,k=p,\varphi$) and $\sqrt{g}=p\,(1+e\cos\varphi)^{-2}$ ([13], eq. (A.5)) is the Jacobian of the transformation from Cartesian coordinates (x, y) to curvilinear coordinates (p, φ). Finally, from the continuity equation for pure Keplerian motion Lyubarskij et al. [13] have derived the useful result that the surface density Σ factorizes into some unspecified function $f(p, e)$ and a known dependence on φ:

(13) $$\Sigma(p,\varphi,e)=\frac{f(p,e)}{\sqrt{g}\,V^{\varphi}},$$

where $V^{\varphi}=\sqrt{GM/p^3}\,(1+e\cos\varphi)^2$ is the contravariant φ-component of the Keplerian velocity ([13], eq. (A.11)). As mentioned in [16], in the case of constant eccentricity disc $\sqrt{g}V^{\varphi}=\sqrt{GM/p}$, i.e., in curvilinear coordinates (p, φ) Σ depends explicitely only on p and e but not on φ [9, 13, 14]. It should be also noted that the energy balance equation is not required to be written in order to close our set of equations because it is already identically fulfilled [13]. Our purpose in this paper is to obtain in an explicit form analytical expressions for $f(p, e)$ and the other accretion disc characteristics (averaged over the disc thickness): temperature T, density ρ, viscosity η, etc. Generally speaking, for a given mass accretion rate $\dot{M}$ and mass of the compact object M, they are expected to be functions of p, φ and e.

III. Solutions to the accretion disc structure equations

The standard α-disc model [1] deals with vertically averaged astrophysical quantities and this approximation is also retained in the considered here viscous elliptical discs [9, 13, 14]. The difference between these two cases is that in the later all quantities are allowed to depend not only on r, but also on φ. As concerns to the α-parameter, we assume the simplest possibility α=constant throughout the disc. Taking into account (9), the ratio $\Sigma/H=\rho(z=0)$ and (2), we obtain

$$K\rho^{\gamma-1}=\frac{1}{8}\omega_K^2 H^2. \tag{14}$$

On the other hand, from (13) follows

$$\rho^{\gamma-1}=(\Sigma/H)^{\gamma-1}=\left[f/\left(H\sqrt{g}\,V^{\varphi}\right)\right]^{\gamma-1}. \tag{15}$$

Combining (14) and (15), and having in mind that $\sqrt{g}\,V^{\varphi}=\sqrt{GM/p}$, we can express the disc thickness H, volume density ρ and pressure P by means of the unknown yet functions K (specifying the gas entropy) and f (specifying the disc surface density Σ):

$$H(p,\varphi,e,K,f)=\frac{1}{\sqrt{GM}}\left[8Kf^{\gamma-1}p^{(\gamma+5)/2}(1+e\cos\varphi)^{-3}\right]^{1/(\gamma+1)}, \tag{16}$$

$$\rho(p,\varphi,e,K,f)=\frac{\Sigma}{H}=\left[\frac{f^2(1+e\cos\varphi)^3}{8Kp^2}\right]^{1/(\gamma+1)}, \tag{17}$$

$$P(p,\varphi,e,K,f)=K^{1/(\gamma+1)}\left[\frac{f^2(1+e\cos\varphi)^3}{8p^2}\right]^{\gamma/(\gamma+1)}. \tag{18}$$

With the above results it is easy to find the viscosity coefficient, integrated over the disc thickness ([13], eq. (48)):

$$\eta(p,\varphi,e,K,f)=\alpha\Sigma v_s H \tag{19}$$
$$=\sqrt{\frac{\gamma}{8}}\,8^{2/(\gamma+1)}\frac{\alpha}{GM}\left[K^2 f^{(3\gamma-1)}p^4(1+e\cos\varphi)^{(3\gamma-9)/2}\right]^{1/(\gamma+1)},$$

where the first equality follows from (9) and $P=\gamma^{-1}\rho v_s^2$ (with v_s the adiabatic speed of sound), analogously to the derivation of eq. (52) in [13]. In the case of elliptical discs with a constant eccentricity (i.e., e_p=0), the convolution of the shear tensor $\sigma_{ik}\,\sigma^{ik}$ may be computed from eq. (A.16) given by Lyubarskij et al. [13]. Note that an additional multiplier $(1+e\cos\varphi)$ may be extracted from the expression in the curly brackets:

$$
\begin{aligned}
(20)\qquad \sigma^2 \equiv \sigma_{ik}\sigma^{ik} &= (GM/2p^3)(1+e\cos\varphi)^{-1}\left[9-2e^2+e^4+\left(33e-2e^3+e^5\right)\cos\varphi\right.\\
&\quad \left.+48e^2\cos^2\varphi+32e^3\cos^3\varphi+8e^4\cos^4\varphi\right]\\
&= \left(GM/2p^3\right)\left[(1-e^2)^2+8(1+e\cos\varphi)^3\right],\ (\text{for } e_p=0).
\end{aligned}
$$

According to (19) and (20), the left hand side of thermal balance equation (12) becomes

$$
\begin{aligned}
(21)\qquad & \frac{1}{2}\int_0^{2\pi}\eta\sigma^2\sqrt{g}\,\mathrm{d}\varphi\\
&= \begin{cases}
\dfrac{2^{-\frac{1}{14}}}{\sqrt{3}}\alpha K^{\frac{6}{7}} f^{\frac{9}{7}} p^{-\frac{2}{7}}\left[(1-e^2)^2\displaystyle\int_0^{2\pi}\frac{\mathrm{d}\varphi}{(1+e\cos\varphi)^{43/14}}+\int_0^{2\pi}\frac{8\mathrm{d}\varphi}{(1+e\cos\varphi)^{1/14}}\right],\\[2ex]
\dfrac{\sqrt{5}}{\sqrt{3}}\dfrac{\alpha}{2^{5/4}}K^{\frac{3}{4}} f^{\frac{3}{2}} p^{-\frac{1}{2}}\left[(1-e^2)^2\displaystyle\int_0^{2\pi}\frac{\mathrm{d}\varphi}{(1+e\cos\varphi)^{11/14}}+\int_0^{2\pi}8(1+e\cos\varphi)^{1/4}d\varphi\right],
\end{cases}
\end{aligned}
$$

were the upper expression holds for $\gamma=4/3$ and the lower refers to $\gamma=5/3$, respectively. Let us now compute the right hand side of (12) for each of the three zones A, B and C. According to (4), valid for a radiation-dominated plasma (zone A : $\gamma=4/3$; $k=k_T$):

$$
\begin{aligned}
(22)\qquad \int_0^{2\pi}\frac{32\sigma_B T^4}{3k\rho H}\sqrt{g}\,\mathrm{d}\varphi &= \frac{8c}{k_T}\int_0^{2\pi}\frac{P}{\Sigma}\sqrt{g}\,\mathrm{d}\varphi\\
&= 2^{9/7}\frac{c\sqrt{GM}}{k_T}K^{3/7}f^{1/7}p^{-9/14}\int_0^{2\pi}\frac{d\varphi}{(1+e\cos\varphi)^{2/7}}.
\end{aligned}
$$

For zone B (ideal gas, $\gamma=5/3$; $k=k_T$), instead of equation (4), now we must use equation (5), which signifies that $T^4=(\mu P/R\rho)^4$ and, correspondingly:

$$
\begin{aligned}
(23)\qquad \int_0^{2\pi}\frac{32\sigma_B T^4}{3k\rho H}\sqrt{g}\,\mathrm{d}\varphi &= \frac{32\sigma_B\mu^4}{3k_T R^4}\int_0^{2\pi}\frac{P^4}{\rho^4\Sigma}\sqrt{g}\,\mathrm{d}\varphi\\
&= \frac{8\pi\sigma_B\mu^4\sqrt{GM}}{3k_T R^4}K^3 f p^{-3/2}.
\end{aligned}
$$

Finally, for the outermost zone C (ideal gas, $\gamma=5/3$; $k=k_{ff}$) we shall take into account (5) and (6) to obtain

$$
\begin{aligned}
(24)\qquad \int_0^{2\pi}\frac{32\sigma_B T^4}{3k\rho H}\sqrt{g}\,\mathrm{d}\varphi &= \frac{32\sigma_B}{3\zeta}\int_0^{2\pi}\frac{T^{15/2}}{\rho\Sigma}\sqrt{g}\,\mathrm{d}\varphi\\
&= \frac{\sqrt{2}\sigma_B\mu^{15/2}\sqrt{GM}}{3\zeta R^{15/2}}K^6 f^2 p^{-5/2}\int_0^{2\pi}(1+e\cos\varphi)^{5/2}d\varphi.
\end{aligned}
$$

In the derivation of (22) - (24) we have again used the representations (13), (16) - (18) for Σ, H, ρ and P, respectively.

The integrals over the angular variable φ can be evaluated by means of the hypergeometric function $F(a, b, c; z)$ [17, 18]:

$$\int_0^{2\pi} \frac{d\varphi}{(1+e\cos\varphi)^x} = 2\pi F\left(\frac{x}{2}, \frac{x}{2}+\frac{1}{2}, 1; e^2\right). \tag{25}$$

The above expression is a converged series, because the eccentricity e of the ellipse is allways less than unity (i.e., $e^2<1$). For integer $x=0, \pm 1, \pm 2, \ldots$, the integrals in the left hand side of (25) can be expressed through Legendre polynomials [18]. In view of the results (21) - (24), thermal balance equation (12) enables us to elliminate the unknown function $K(p, \varphi, e)$:

$$K_A^{6/7} = A(e) f^{-16/7} p^{-5/7}, \text{ (for zone A)}, \tag{26}$$

$$A(e) \equiv \left(\frac{\sqrt{3}\, 2^{\frac{17}{14}} c\sqrt{GM}}{\alpha k_T} \right)^2 \left[\frac{F\left(\frac{1}{7}, \frac{9}{14}, 1; e^2\right)}{(1-e^2)^2 F\left(\frac{43}{28}, \frac{57}{28}, 1; e^2\right) + 8F\left(\frac{1}{28}, \frac{15}{28}, 1; e^2\right)} \right]^2,$$

$$K_B^{3/4} = B(e) f^{1/6} p^{1/3}, \text{ (for zone B)}, \tag{27}$$

$$B(e) \equiv \left(\frac{\sqrt{15}\, \alpha k_T R^4}{2^{\frac{13}{4}} \sqrt{GM}\, \sigma_B \mu^4} \right)^{\frac{1}{3}} \left[(1-e^2)^2 F\left(\frac{11}{8}, \frac{15}{8}, 1; e^2\right) + 8F\left(-\frac{1}{8}, \frac{3}{8}, 1; e^2\right) \right]^{\frac{1}{3}},$$

$$K_C^{3/4} = C(e) f^{-1/14} p^{2/7}, \text{ (for zone C)}, \tag{28}$$

$$C(e) \equiv \left(\frac{\sqrt{15}\, \alpha \zeta R^{15/2}}{2^{\frac{7}{4}} \sqrt{GM}\, \sigma_B \mu^{\frac{15}{2}}} \right)^{\frac{1}{7}} \left[\frac{(1-e^2)^2 F\left(\frac{11}{8}, \frac{15}{8}, 1; e^2\right) + 8F\left(-\frac{1}{8}, \frac{3}{8}, 1; e^2\right)}{F\left(-\frac{5}{4}, -\frac{3}{4}, 1; e^2\right)} \right]^{\frac{1}{7}},$$

where $A(e)$, $B(e)$ and $C(e)$ are known functions of the constant (over the whole accretion disc) eccentricity e. With the above results, we can now return to the expression (19) for the viscosity coefficient $\eta\,(p, \varphi, e, f)$ in which we set appropriate values for γ ($\gamma=4/3$ in zone A; $\gamma=5/3$ in zones B and C):

$$\eta(p,\varphi,e,f) = \begin{cases} 2^{\frac{29}{14}} \dfrac{\alpha A(e)}{\sqrt{3}\, GM} f^{-1} p\, (1+e\cos\varphi)^{-15/14}, & \text{(zone A)}, \\ 2^{\frac{3}{4}} \dfrac{\sqrt{5}\, \alpha B(e)}{\sqrt{3}\, GM} f^{\frac{5}{3}} p^{\frac{11}{6}} (1+e\cos\varphi)^{-3/4}, & \text{(zone B)}, \\ 2^{\frac{3}{4}} \dfrac{\sqrt{5}\, \alpha C(e)}{\sqrt{3}\, GM} f^{\frac{10}{7}} p^{\frac{25}{14}} (1+e\cos\varphi)^{-3/4}, & \text{(zone C)}. \end{cases} \tag{29}$$

The auxiliary function $Y(e, \varphi)$ is already computed for constant eccentricity discs (8). Consequently, we are now in a position to evaluate the left hand side $\int_0^{2\pi} \eta Y d\varphi$ of the angular momentum balance (7). The later becomes an equation only for one unknown function $f(p, e)$ which can easily be solved, using again hypergeometric functions (25). We stress that the integration constants $D_A(e)$, $D_B(e)$ and $D_C(e)$ (for each zone A, B and C) in (7) may be, generally speaking, different for these three distinct regions. This situation arises from the possibility to choose specific boundary conditions (determining $D(e)$) for every zone. How to select these conditions, in order to have a self-consistent global structure of the accretion disc, is beyond the scope of this paper. We assume a priori that our investigation concerns only with such gas particles which streamlines fall well inside the considered zone and do not approach very close its boundaries. Solving the equation of angular momentum balance (7), we have:

$$(30) \qquad f_A(p,e) = \tilde{A}(e) \frac{p}{\dot{M} - D_A(e)/\sqrt{GMp}} \approx \frac{\tilde{A}(e)}{\dot{M}} p, \qquad \text{(zone A)},$$

$$\tilde{A}(e) \equiv \frac{2^{\frac{29}{14}} \pi \alpha A(e)}{\sqrt{3}\, GM(1-e^2)^{11/7}} \left[4F\left(-\frac{1}{28}, -\frac{15}{28}, 1; e^2\right) - F\left(-\frac{15}{28}, -\frac{29}{28}, 1; e^2\right) \right]$$

$$(31) \qquad f_B(p,e) = \frac{1}{\tilde{B}(e)} \left[\dot{M} - \frac{D_B(e)}{\sqrt{GMp}} \right]^{\frac{3}{5}} p^{-\frac{11}{10}} \approx \frac{\dot{M}}{\tilde{B}(e)} p^{-\frac{11}{10}}, \qquad \text{(zone B)},$$

$$\tilde{B}(e) \equiv \left[2^{\frac{3}{4}} \frac{\sqrt{5}\, \pi \alpha B(e)}{\sqrt{3}\, GM} \right]^{\frac{3}{5}} (1-e^2)^{-\frac{3}{4}} \left[4F\left(\frac{1}{8}, -\frac{3}{8}, 1; e^2\right) - F\left(-\frac{3}{8}, -\frac{7}{8}, 1; e^2\right) \right]^{\frac{3}{5}},$$

$$(32) \qquad f_C(p,e) = \frac{1}{\tilde{C}(e)} \left[\dot{M} - \frac{D_C(e)}{\sqrt{GMp}} \right]^{\frac{7}{10}} p^{-\frac{5}{4}} \approx \frac{\dot{M}}{\tilde{C}(e)} p^{-\frac{5}{4}}, \qquad \text{(zone C)},$$

$$\tilde{C}(e) \equiv \left[2^{\frac{3}{4}} \frac{\sqrt{5}\, \pi \alpha C(e)}{\sqrt{3}\, GM} \right]^{\frac{7}{10}} (1-e^2)^{-\frac{7}{8}} \left[4F\left(\frac{1}{8}, -\frac{3}{8}, 1; e^2\right) - F\left(-\frac{3}{8}, -\frac{7}{8}, 1; e^2\right) \right]^{\frac{7}{10}},$$

where the approximate equalities are referenced to the conditions $p >> D_i^2 / GM\dot{M}^2$, $i = A, B, C$.

IV. Discussion and conclusions

The last three relations (30) - (32) close the solution of the problem, because the coefficients $A(e), \tilde{A}(e), \ldots, \tilde{C}(e)$ are known functions of the eccentricity e. For computational reasons, it is preferable hypergeometric functions $F(a, b, c; z)$ to have arguments which are less or at least of order of unity (i.e., in our case we want $|a| \lesssim 1$, $|b| \lesssim 1$) This condition insures their high accuracy computation even with mini-calculators [19]. It is useful to apply the linear transformation [18, 20]

$$(33) \qquad F(a,b,c;z) = (1-z)^{c-a-b} F(c-a, c-b, c; z).$$

For example, computing $A(e)$ (26), we may set $(1-e^2)^{-4/7} \times F\left(-\frac{15}{28}, -\frac{29}{28}, 1; e^2\right)$ instead of $(1-e^2)^2 F\left(\frac{43}{28}, \frac{57}{28}, 1; e^2\right)$, etc.

Of course, the integration constants $D_A(e)$, $D_B(e)$, and $D_C(e)$, still remain undetermined (if their values cannot be neglected in the limit of large p). They may be utilized as additional degrees of freedom to glue more flexibly the three zones of the considered α-disc model. In particular, it is possible to impose the condition of continuity of the surface density $\Sigma(p, \varphi, e)$ and its derivative $\frac{\partial}{\partial p}\Sigma(p, \varphi, e)$ during the transition from one zone to another neighbouring one.

We shall write in an explicite form the final analytical expressions for the entropy parameter K and disc surface density Σ (in which it is evident that the angular dependence on φ is absent, i.e., K and Σ are streamline functions only):

$$(34) \qquad K(p,e) = \begin{cases} A^{7/6}(e)\tilde{A}^{-8/3}(e)\left[\dot{M} - \dfrac{D_A(e)}{\sqrt{GMp}}\right]^{8/3} p^{-7/2}, & \text{(zone A)}, \\ B^{4/3}(e)\tilde{B}^{-2/9}(e)\left[\dot{M} - \dfrac{D_B(e)}{\sqrt{GMp}}\right]^{2/15} p^{1/5}, & \text{(zone B)}, \\ C^{4/3}(e)\tilde{C}^{2/21}(e)\left[\dot{M} - \dfrac{D_C(e)}{\sqrt{GMp}}\right]^{-1/15} p^{1/2}, & \text{(zone C)}, \end{cases}$$

$$(35) \qquad \Sigma(p,e) = \begin{cases} \dfrac{\tilde{A}(e)}{\sqrt{GM}} \dfrac{p^{3/2}}{\dot{M} - D_A(e)/\sqrt{GMp}}, & \text{(zone A)}, \\ \dfrac{1}{\tilde{B}(e)\sqrt{GM}}\left[\dot{M} - D_B(e)/\sqrt{GMp}\right]^{3/5} p^{-3/5}, & \text{(zone B)}, \\ \dfrac{1}{\tilde{C}(e)\sqrt{GM}}\left[\dot{M} - D_C(e)/\sqrt{GMp}\right]^{7/10} p^{-3/4}, & \text{(zone C)}. \end{cases}$$

The other structure parameters of the accretion disc may be found by substituting (30) - (32) and (34) into (16) - (19).

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Елиптични акреционни дискове с постоянен ексцентрицитет. II. Стандартен модел на α-диск

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(Резюме)

Получени са аналитични изрази за физическите условия (налягане, температура, плътност и т.н.), характеризиращи вертикално усреднената структура на акреционни дискове с постоянен ексцентрицитет. Решената система от уравнения включва уравнението на непрекъснатостта, баланса на ъгловия момент, хидростатичното равновесие и топлинния баланс. Тези решения са получени за всяка една от трите зони (вътрешна, средна и външна) на стандартния модел на α-диск, обобщен от Сиер и Кларк и от Любарски и др. Нашето изследване третира само стационарна акреция и не установява граничните условия, определящи константите на интегриране. Показано е, че ако последните са пренебрежими, то повърхностната плътност и коефициентът на ентропия в уравнението на адиабатата се разлагат на произведение от известни функции, зависещи поотделно от фокалния параметър p и ексцентрицитета e. Отбелязано е сходството в поведението по дължината на радиуса между решенията за елиптични и кръгови дискове.