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Elliptical accretion discs with constant eccentricity. I. Case $\eta = \beta\Sigma^n$

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I. Introduction

In the majority of papers which deal with the accretion discs, it is a priori assumed that the disc particles are moving around the primary star on circular orbits [1, 2]. Nevertheless, there have been performed theoretical investigations, indicating that the circular motion is not the only possibility for description of the disc dynamics. Numerical simulations of Whitehurst and King [3, 4] have shown development of stable eccentric accretion discs in the close binary systems as a result of the tidal influence of the secondary star. As indicate their researches, the outer parts of the discs are more elongated than the inner regions. This is naturally explained by the greater value of the tidal force at large disc radii. Analytical searches of Syer and Clarke [5, 6] and Lyubarskij et al. [7] are concentrated over the discs for which eccentricity e does not depend on the disc radius r. In particular, Lyubarskij et al. [7] prove that the case e=const may be realized under quite general dependence of the viscosity law on the azimuthal angle φ. It should also be noted that in the later paper stability analysis of the circular nonstationary accretion discs leads to the conclusion that for some viscosity laws (e.g., for α-viscosity) these discs are unstable with respect to the growth of the eccentricity. Consequently, from theoretical grounds, it is not unreasonable to include into our considerations accretion discs with constant eccentricity e.

Observational data from some binary systems also support the possibility that the noncircular accretion discs are realy existing objects in the nature. In particular, tidally distorted eccentric discs with time-dependent sizes are an useful tool for explanation of the superhump period of the light-curves of SU UMa, VW Hyi, TU Men and some other binary stars [8, 9]. Although a nonviscous accretion also may produce discs with nonaxisymmetric surface density distribution Σ [10, 11], throughout this paper we shall limit us to viscosity governed accretion discs.

II. Accretion disc model

Let us consider a geometrically thin Keplerian accretion disc. The trajectories of the fluid elements (i.e., streamlines) are assumed to be cofocal ellipses which are described by two quantities: the eccentricity e and the focal parameter p. The origin of the coordinate system coincides with the centre of the primary start around which the disc particles slowly spiral in by nearly Keplerian orbits. We restrict our treatment of the problem to the simple case in which the apse lines of all orbits are in line with each other, i.e., the semimajor axis of the ellipses lie on the abscissa. Consequently, in polar coordinates (r, φ) the position of any particle is given by

(1) $$r = p/(1 + e\cos\varphi).$$

The evolution of the accretion disc may be determined from the energy and angular momentum exchange between adjacent fluid contours caused by the viscosity. It should be emphasized that unlike the elliptical orbits in selestial mechanics, self-intersecting adjacent Keplerian orbits in gaseous discs are excluded from considerations. Another important feature is that the focal parameter p and the eccentricity e are not, generally speaking, independent variables of the accretion disc model. For example, Lyubarskij et al. [7], solving the problem of stationary viscous accretion in eccentric discs, have obtained a second order differential equation for $e=e\ (p)$. Solving it numerically, they conclude that three classes of solutions exist (see Figs. 2, 4 and 5 from [7]). In particular, it is evident that the solution e=const is available – a result which is also confirmed analitically by Lyubarskij et al. [7]. The most attractive property of the accretion discs with an arbitrary constant eccentricity of the orbiting particles is the possibility to exist for any reasonable viscosity law dependence on the azimuthal angle φ. Consequently, there is not need to specify viscosity in details when we are dealing with constant eccentricity discs.

Following the notations of Lyubarskij et al. [7], we shall use as nonorthogonal curvilinear Eulerian coordinates the focal parameter p and the azimuthal angle φ. Since the evolution of the nearly Keplerian accretion discs with time is slow in comparison with the Keplerian time-scale, it is easy to show that in the case of constant e (i.e., $\partial e/\partial p \equiv e_p = 0$) the disc surface density $\Sigma\ (p, \varphi, e, t)$ may be factorized [7]

(2) $$\Sigma\,(p,\varphi,e,t) = f(p,e,t)/\sqrt{g}\,V^{\varphi},$$

where

(3) $$g = p^2/(1 + e\cos\varphi)^4$$

is the determinant of the metric tensor and

(4) $$V^{\varphi} = \left(GM/p^3\right)^{1/2}(1 + e\cos\varphi)^2$$

is the contravariant φ-component of the Keplerian velocity. G and M are the Newton's gravitational constant and the mass of the compact object in the disc centre, respectively. Our purpose in this paper is to obtain in the stationary case $(\partial f/\partial t = \partial\Sigma/\partial t = 0)$ an explicit expression for the unknown function $f\ (p, e)$ using an appropriate viscosity law. In their investigation Lyubarskij et al. [7] introduce an auxiliary function $Y\ (e, \varphi)$ through the relation

(5) $$g r^p \sigma^{p\varphi} = -\frac{3}{2}(GMp)^{1/2}\,Y(e,\varphi),$$

where $r^p=p$ is the contravariant p-component of the radius vector $\bar{r}$ (for e_p=0) and

(6) $$\sigma^{p\varphi} = -\frac{3}{2}\left(\frac{GM}{p^5}\right)^{1/2}(1 + e\cos\varphi)\left[1 + \frac{7}{3}e\cos\varphi + \frac{1}{3}e^2\left(1 + 4\cos^2\varphi\right) + \frac{1}{3}e^3\cos\varphi\right],$$

(for e_p=0) is the contravariant $p\varphi$-component of the shear tensor σ^{ik} (i, k=p, φ).

Taking into account equalities (5) and (6), it is easy to compute

$$Y\left(e_p=0,\varphi\right)=\frac{1}{3}(1+e\cos\varphi)^{-3}\left(3+e^2+7e\cos\varphi+4e^2\cos^2\varphi+e^3\cos\varphi\right). \tag{7}$$

In the model under consideration, the disc evolution is determined only by the viscous stresses $w^{ik}=\eta\sigma^{ik}(i,\ k=p,\ \varphi)$, where η is the intergrated over the disc thickness viscosity coefficient. It should be noted that according to Syer and Clarke's results [6, 12], if the disc viscosity coefficient η is a separate function of streamline and azimuthal angle φ, the rate of change of the eccentricity e with the orbital phase $\partial e/\partial\varphi \equiv e_\varphi$ can be expressed as a perfect differential. From this follows that orbit averaged value $< e_\varphi >$ equals to zero and the later authors conclude that, to first order of the perturbation theory, the viscous elliptical discs exactly preserve their eccentricity during the accretion process. In other words, the elliptical discs, once formed in any way in a Keplerian potential, are long living structures and the stationary approximation can not be ad hoc excluded from considerations as an unreal description of the accretion.

III. Power law viscosity

We shall consider the model case of an $\eta=\beta\Sigma^n$ viscosity law, where the multiplier β and the power n ($0.5 \le n \le 3.0$) have constant values. Integrating over p the equation of angular momentum balance, Lyubarskij et al. [7] have obtained for the case $e_p=0$ the following relation between the shear viscosity coefficient η and the mass accretion rate $\dot M$:

$$\int_0^{2\pi}\eta(p,\varphi)\,Y(e,\varphi)\,d\varphi=\frac{2}{3}\left[\dot M-D(\beta,n,e)/\sqrt{GMp}\,\right], \tag{8}$$

where D (β, n, e) is an integration constant depending on β, n and e, but not on p and φ. In their investigation Lyubarskij et al. [7] have neglected D, limiting their results to these parts of the disc which are far away from its inner boundary, i.e., for p large enough in order to be satisfied $\dot M \gg D/\sqrt{GMp}$. We, however, shall preserve the constant D. Substituting the adopted power law $\eta=\beta\Sigma^n$ into (8) and taking into account (2), we obtain an expression for the unknown function f (p, e), similar to eq. (39) in [7]. It includes an angle averaging of the auxiliary function $Y(e,\ \varphi)$:

$$f^n(p,e)=\frac{2}{3\beta}\left[\dot M-\frac{D(\beta,n,e)}{\sqrt{GMp}}\right]\left[\int_0^{2\pi}\frac{Y(e,\varphi)\,d\varphi}{\left(\sqrt{g}\,V^\varphi\right)^n}\right]^{-1}, \tag{9}$$

where the mass accretion rate $\dot M$ is related to the surface density Σ. In this paper we shall consider $\dot M$ as a constant parameter of the model, having in mind that for a stationary accretion $\dot M$ does not depend on time t and the focal parameter p. First of all, we note that in the case e=const throughout the disc, $\left(\sqrt{g}V^\varphi\right)^n=(GM/p)^{n/2}$ does not include dependence on φ and we have to evaluate only the intergrated over φ value of the auxiliary function Y (e, φ). Secondly, we have exact analytical expressions about the following integrals [13, 14]:

$$\int_0^{2\pi}(1+e\cos\varphi)^{-3}\,d\varphi=\pi\left(2+e^2\right)\left(1-e^2\right)^{-5/2}, \tag{10}$$

$$\int_0^{2\pi}\cos\varphi\,(1+e\cos\varphi)^{-3}\,d\varphi=-3\pi e\left(1-e^2\right)^{-5/2}, \tag{11}$$

$$\int_0^{2\pi} \cos^2\varphi\,(1+e\cos\varphi)^{-3}\,d\varphi = \pi\left(1+2e^2\right)\left(1-e^2\right)^{-5/2}. \tag{12}$$

Finally, according to the above relations, we simply have for $e_p=0$ $\int_0^{2\pi} Y(e,\varphi)\,d\varphi = 2\pi\left(1-e^2\right)^{-1/2}$. Correspondingly:

$$f(\beta,n,p,e)=\left(\frac{GM}{p}\right)^{1/2}\left\{\frac{\sqrt{1-e^2}}{3\pi\beta}\left[\dot{M}-\frac{D(\beta,n,e)}{\sqrt{GMp}}\right]\right\}^{1/n}, \tag{13}$$

$$\eta(\beta,n,p,e)=\frac{\sqrt{1-e^2}}{3\pi}\left[\dot{M}-\frac{D(\beta,n,e)}{\sqrt{GMp}}\right] \tag{14}$$

$$\Sigma(\beta,n,p,e)=\left\{\frac{\sqrt{1-e^2}}{3\pi\beta}\left[\dot{M}-\frac{D(\beta,n,e)}{\sqrt{GMp}}\right]\right\}^{1/n}. \tag{15}$$

These expressions illustrate the well known result that for constant eccentricity elliptical discs the viscosity coefficient η and the surface density Σ are functions on streamlines only [5–7]. In the outer parts of the disc (for $p \gg D^2(\beta,n,e)/GM\dot{M}^2$) η and Σ become independent also on p and approach constant values:

$$f(\beta,n,p,e)\approx\sqrt{GM/p}\left(\dot{M}\sqrt{1-e^2}/3\pi\beta\right)^{1/n}, \tag{16}$$

$$\eta(\beta,n,p,e)\approx\eta_{\max}(e)\equiv\dot{M}\sqrt{1-e^2}/3\pi, \tag{17}$$

$$\Sigma(\beta,n,p,e)\approx\Sigma_{\max}(\beta,n,e)\equiv\left(\dot{M}\sqrt{1-e^2}/3\pi\beta\right)^{1/n}. \tag{18}$$

For stationary circular accretion discs (e=0, p=r), (16) and (17) transform into the expressions given by Lyubarskij et al. [7]: $f_0(\beta,n,p)=\sqrt{\frac{GM}{p}}\left(\frac{\dot{M}_0}{3\pi\beta}\right)^{1/n}$ and $\eta_0=\dot{M}_0/3\pi$.

IV. Boundary conditions

In order to evaluate the integration constant $D(\beta, n, e)$ it is appropriate to utilize any physically reasonable conditions at the inner boundary of the accretion disc. Our investigation deals with the stationary case and, consequently, these conditions would also be independent on time t. Determination of such boundary conditions may be strongly complicated if general relativistic effects must be taken into account. But in the later case, as mentioned by Syer and Clarke [5, 12], differential precession leads to a circularization of the inner parts of the disc. For this reason we would expect that our supposition of constant eccentricity orbits with $e \neq 0$ throughout the disc is not strictly fulfilled. Nevertheless, we shall simply limit us to the Newtonian mechanics description of the inner disc. But even in that case, the presence of the secondary star may cause a differential precession because of deviations from a Keplerian potential [5, 11]. We also neglect this possibility and assume that giving not very precisely the inner boundary conditions, the global disc structure would not be drastically affected.

Following the suggestion of Shakura and Sunyaev [15], we may suppose that the viscous stress is nearly equal to zero on the last stable orbit with $p = p_{min}$, i.e., $\eta(p_{min}, e) \approx 0$. This condition enables us to evaluate the integration constant $D(\beta, n, e)$

$$D(\beta,n,e) = \dot{M}\sqrt{GM\,p_{min}} \tag{19}$$

through the minimal value of the focal parameter p_{min}. It should be noted that in the particular case of circular orbits $p_{min}=3R_g$ for a nonrotating black hole or a neutron star with a radius $R_s<3R_g$, where R_g is the Schwarzschild gravitational radius. During the transition from $p>p_{min}$ to $p<p_{min}$ the character of the gas particles motion abruptly changes. The nearly Keplerian orbiting with a slow radial drift is transformed into a fast radial falling without energy release due to viscous forces. Similar situation is expected for elliptical orbits if $r_{min} = \dfrac{p_{min}}{1+e} < 3R_g$, where the transition from one type orbits to another type occures at the pericentre. According to (17) – (19), the expressions for viscosity and disc surface density can be rewritten in the following form:

$$\eta(p,e) = \frac{\dot{M}\sqrt{1-e^2}}{3\pi}\left[1-\left(\frac{p}{p_{min}}\right)^{-1/2}\right] = \eta_{max}(e)\left[1-\left(\frac{p}{p_{min}}\right)^{-1/2}\right], \tag{20}$$

$$\Sigma(p,e) = \left(\frac{p}{GM}\right)^{1/2} f(p,e) = \left(\frac{\dot{M}\sqrt{1-e^2}}{3\pi\beta}\right)^{1/n}\left[1-\left(\frac{p}{p_{min}}\right)^{-1/2}\right]^{1/n} \tag{21}$$

$$= \Sigma_{max}(e)\left[1-\left(\frac{p}{p_{min}}\right)^{-1/2}\right]^{1/n}.$$

Obviously, the vanishing of the viscosity $\eta\,(p_{min}, e)=0$ at the inner disc edge implies also vanishing of the disc surface density $\Sigma(p_{min}, e)=0$ at the same place because of the a priori accepted viscosity law $\eta=\beta\Sigma^n$. Sometimes it is more appropriate to decide that Σ does not approach zero value at $p=p_{min}$. For example, if $R_s>3R_g$ or the compact object (primary star) is a white dwarf, a boundary layer is expected to exist between the accretion disc and the stellar surface. Correspondingly, the disc surface density Σ will not drop to zero at the inner boundary. Moreover, as we have mentioned earlier, at the innermost region of the disc a circularization of the particle orbits may occur. Then the assumption e=const throughout the disc is not a reasonable approximation for all disc radii. Consequently, it may be preferable to choose p_{min} to be such a value of the focal parameter p, above which our description of the accretion disc structure (with $e_p=0$) is valid. This situation suggests that it may be more appropriate boundary conditions of the considered model to be given at some value $p=p_{min}$ which differs from the focal parameter at the inner geometrical boundary of the disc. We are able to modify (20) – (21) for the case of nonzero density $\Sigma_{min}(e) \equiv \Sigma(p_{min}, e) \neq 0$. The result is a slightly more general form of these expressions:

$$\eta(p,e) = \eta_{\max}(e)\left\{1 - \left[1 - \left(\Sigma_{\min}(e)/\Sigma_{\max}(e)\right)^n\right]\left(\frac{p}{p_{\min}}\right)^{-1/2}\right\}, \tag{22}$$

$$\Sigma(p,e) = \left(\frac{p}{GM}\right)^{1/2} f(p,e) \tag{23}$$

$$= \Sigma_{\max}(e)\left\{1 - \left[1 - \left(\Sigma_{\min}(e)/\Sigma_{\max}(e)\right)^n\right]\left(\frac{p}{p_{\min}}\right)^{-1/2}\right\}^{1/n}.$$

Therefore, the problem of determining of the integration constant $D(\beta, n, e)$ in the angular momentum balance is transformed into the problem of finding of an appropriate value of the accretion disc surface density Σ around a given streamline. This may be more useful situation when the results (13) – (15) are applied to concrete constant eccentricity disc models.

V. Discussion and conclusions

Our consideration of elliptical accretion discs is limited to the case of nested cofocal constant eccentricity streamlines. We have also adopted the approximation of a power law viscosity dependence $\eta = \beta\Sigma^n$, where parameters β and n are assumed to have constant values throughout the disc. We have derived in an explicit form analytical expressions for the surface density Σ and the integrated over the disc thickness shear viscosity coefficient η. According to conclusions of Syer and Clarke [5, 6], Σ and η are functions of streamlines only. Even more, for the outer disc regions $\left(p >> D^2(\beta,n,e)/GM\dot{M}^2\right)$ these quantities approach constant values. Formulae (20) – (23) can easily be rewritten in the usual polar coordinates (r, φ) if the focal parameter p is replaced by means of (1): $p = r(1 + e\cos\varphi)$. Having in mind that the expected values of n lie approximately between 0 and 3, a weak dependence on φ appears in the expressions for $\Sigma(r, \varphi)$ and $\eta(r, \varphi)$. It should be noted that giving of the disc boundary conditions at its inner part may possibly strike with difficulties (like the Lightman – Eardley instability [16]), in addition to the other approaches which limit the application of (20) – (23) to the real accretion discs. For example, the disc may not have a mirror symmetry with respect to the direction of pericentre–apocentre, which in our treatment is assumed to be the same for all ellipces. As pointed out by Syer and Clarke [5], when the eccentricity exceeds a critical value, the flow is relatively thickened for a section of the flow downstream of apocentre and, as a consequence, a prograde precession of the streamline will follow. They have also mentioned that it is not clear at present time whether this mechanism of circularization works for all parts of the accretion disc.

Nowadays it is not doubtfull that elliptical accretion discs may exist at least around some of the known compact objects, preferably in close binary stellar systems. The analytical expressions derived in this paper would be usefull for computation of the theoretical disc brightness distributions and profiles of disc spectral lines. The later seem highly asymmetric depending on the disc eccentricity e and the disc inclination to the observer's line of sight [5, 9]. Changes of the accretion disc characteristics during the orbital motion will, in principle, give additional possibilities for verification of the approaches made in the considered model, e.g., constant eccentricity e along the disc radius, stationarity, averaging of the disc parameters over its height, etc.

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Елиптични акреционни дискове с постоянен ексцентрицитет. I. Случай $\eta = \beta\Sigma^n$

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(Резюме)

Получени са точни аналитични изрази за повърхностната плътност Σ и коефициента на сдвиговия вискозитет η за елиптични акреционни дискове с постоянен ексцентрицитет e на кофокалните токови линии на

частиците. Предположена е априорно степенната зависимост $\eta = \beta\Sigma^n$ с постоянни параметри β и n. Във външните части на диска Σ и η се стремят към постоянни значения, зависещи от скоростта на акреция на вещество $\dot{M}, \beta, n$ и e.

Най-вътрешната област на диска се характеризира с бавно намаляващи Σ и η. В последния случай е въведен един допълнителен параметър (като едно гранично условие) $-\Sigma_{min}$, който задава повърхностната плътност при минимално значение на фокалния параметър p_{min}. Разгледаният модел на акреционен диск е валиден при допускането на стационарна акреция $\dot{M} = \text{const}$.