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Optimal control of unpowered flight in vertical plane[1]

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1. Introduction

The way the energy of gliding aircraft is used may contribute to an increase in their effectiveness, and in some cases, i.e. a flight of an aircraft with engine failure, directly relates to the probability of its survival. Many of the traditional methods for control of unpowered flight are empirically derived, which makes them suitable in a limited range of initial conditions. The total energy of an aircraft would be effectively used only through implementation of optimal control strategies or their close approximations.

Methods for trajectory optimization are under constant development. A recent example is the attempt for direct trajectory optimization via representation of the dynamical system in differential inclusion format [1]. Nevertheless, most successful solutions of the problem are reached through numerical implementation of the Pontryagin's minimum principle. Unfortunately, many of the solutions are based on strong assumptions of trajectory segments in the vertical flight path [2,3], fixed flight time [1,2] and quite arbitrary definition of boundary conditions [2,3].

In the current paper, the optimal control problem is solved via Pontryagin's minimum principle. The flight time is not fixed. The boundary conditions have physical meaning. The problem of optimal control of unpowered flight is solved according to three criteria:

- Maximum flight range;
- Maximum kinetic energy at the point of impact with the Earth surface;
- Optimal conditions for surface penetration in the point of impact.

A penalty function approach is used to account for end-state constraints. The two-point boundary value problem is solved numerically. The same procedure is used for the three criteria. The differences are in the formulae for the end values of the costate variables.

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The optimal control of the unpowered flight guarantees effective use of the potential and kinetic energy of the aircraft. Its implementation would substantially increase the flight range, the aircraft energy at the point of impact, and the penetration characteristics. Furthermore, it would provide opportunities to compensate for model parameter changes and perturbations.

2. Problem formulation

2.1. Equations of motion

The equations of motion relative to an Earth-fixed coordinate system were described using a rigid-body dynamical model in absence of wind.

$$\text{(1)} \quad \begin{cases} \dot{v} = -\dfrac{D}{m} - g\sin\gamma, \\ \dot{\gamma} = \dfrac{L}{mv} - \dfrac{g\cos\gamma}{v}, \\ \dot{x} = v\cos\gamma, \\ \dot{h} = v\sin\gamma \end{cases}$$

Here m denotes the aircraft mass, and g – the gravitational acceleration. The state variables are speed v, vertical flight path angle γ, horizontal flight distance x, and altitude h. The aircraft drag D and lift L are described by the equations

$$\text{(2)} \quad D = c_D \frac{\rho v^2}{2} S, \qquad L = c_L \frac{\rho v^2}{2} S,$$

where S is the wing area, ρ is the air density, and the parameters c_D and c_L depend on the Mah number M. In the particular model the aerodynamics parameters are defined by the equations

$$\text{(3)} \quad c_D = c_{D_0}(M) + c_L^{\alpha}\alpha^2, \qquad c_L = c_L^{\alpha}\alpha^2,$$

where α is the aircraft angle of attack. Denoting ρ_{N0} as air density at sea level, $H(h)= \rho/\rho_{N0}$ as relative density, and $\mu = \rho_{N0} S/(2m)$, eqs. (1) take the form

$$\text{(4)} \quad \begin{cases} \dot{v} = -c_D \mu H(h) v^2 - g\sin\gamma, \\ \dot{\gamma} = c_L \mu H(h) v - \dfrac{g\cos\gamma}{v}, \\ \dot{x} = v\cos\gamma, \\ \dot{h} = v\sin\gamma \end{cases}$$

The functions $H(h)$ and $M(v,h)$ are described in detail in [4].

2.2. Optimization problem

The optimal control problem is stated in Mayer form and requires minimization of the criterion

$$J = \theta\,[y(t_f),\ t_f] \tag{5}$$

for differential constraints on the state vector $y^T(t) = [v(t),\ \gamma(t),\ x(t),\ h(t)]$ in the form

$$\dot{y} = f(y, \alpha, t), \quad y(t_0) = y_0 \tag{6}$$

described by eqs. [4], constrains on the end state

$$P\,[y(t_f),\ t_f] = 0, \tag{7}$$

and saturation-type constraints on the control function α

$$|\alpha(t)| \le \alpha_s. \tag{8}$$

The duration of the flight t_f is not fixed.

The solution for three different criteria is examined:

a. Maximum flight range

$$J = x\,(t_f); \tag{9a}$$

b. Maximum kinetic energy at the point of impact with the Earth surface

$$J = -0.5\, v^2(t_f); \tag{9b}$$

c. Optimal conditions for surface penetration in the point of impact [5]

$$J = -\,v(t_f)\ \sin\,\gamma(t_f). \tag{9c}$$

The constraints on the end state for the three cases are as follows:

$$h(t_f) = 0, \tag{10a}$$

$$h(t_f) = 0,\ x\,(t_f) - X = 0, \tag{10b, c}$$

where X is the initial horizontal distance to the required point of impact with the Earth surface.

The problem is solved via Pontryagin's minimum principle. The second constraint in eqs. (10*b*,*c*) is accounted for by a penalty function in criteria (9*b*), (9*c*):

$$J = -\,0.5\, v^2(t_f) + 0.5\, s_b\,[x(t_f) - X]^2, \tag{11b}$$

$$J = -\,v\,(t_f)\ \sin\,\gamma\,(t_f) + 0.5\, s_c\,[x(t_f) - X]^2. \tag{11c}$$

The first constraint in eqs. [10] is accounted for by a Lagrange multiplier ψ, which leads to a unconditional minimization of

(12) $$\Pi(t_f) = J[y(t_f), t_f] + \psi h(t_f).$$

The Hamiltonian for the optimal control problem is defined by the equations

(13) $$\mathrm{H}(y,\alpha,\lambda,t) = \lambda^T f,$$

(14) $$\dot{\lambda} = -\frac{\partial \mathrm{H}}{\partial y},$$

(15) $$\dot{\lambda}(t_f) = -\frac{\partial \Pi(t_f)}{\partial y(t_f)}.$$

The Lagrange multiplier and the flight time are determined from eqs. (7) and the condition [6]

(16) $$\mathrm{H}(t_f) + \frac{\partial J[y(t_f), t_f]}{\partial t_f} + \frac{\partial P^T[y(t_f), t_f]}{\partial t_f}\psi = 0.$$

Eqs.(14) for the costate variables have the form

(17) $$\begin{aligned}
\dot{\lambda}_v &= \lambda_v \mu H(h)\left(\frac{dc_{D0}}{dM}\frac{dM}{dv}v^2 + 2c_D v\right) - \lambda_\gamma\left(\mu H(h)c_L + \frac{g\cos\gamma}{v^2}\right) \\
&\quad - \lambda_x \cos\gamma - \lambda_h \sin\gamma, \\
\dot{\lambda}_\gamma &= \lambda_v g\cos\gamma - \lambda_\gamma \frac{g}{v}\sin\gamma + \lambda_x v\sin\gamma - \lambda_h v\cos\gamma, \\
\dot{\lambda}_x &= 0, \\
\dot{\lambda}_h &= \lambda_v \mu v^2\left(\frac{dc_{D0}}{dM}\frac{dM}{dh}H(h) + c_D\frac{dH}{dh}\right) - \lambda_\gamma \mu v c_L \frac{dH}{dh},
\end{aligned}$$

where the dependence of the derivatives dM/dv, dM/dh, and dH/dh on the phase state is derived analytically from the relations in [4]. The end values of the costate variables are defined from eqs.(7), (16), and for the particular criteria are as follows:

(18a) $$\begin{aligned}
\lambda_v(t_f) &= 0, \\
\lambda_\gamma(t_f) &= 0, \\
\lambda_x(t_f) &= 1, \\
\lambda_h(t_f) &= -\frac{1}{\mathrm{tg}\gamma(t_f)}
\end{aligned}$$

$$\lambda_v(t_f)=v(t_f),$$

$$\lambda_\gamma(t_f)=0,$$

(18*b*)
$$\lambda_x(t_f)=-s_b[x(t_f)-X],$$

$$\lambda_h(t_f)=-\frac{\dot{v}(t_f)-s_b[x(t_f)-X]\cos\gamma(t_f)}{\sin\gamma(t_f)};$$

$$\lambda_v(t_f)=-\sin\gamma(t_f),$$

$$\lambda_\gamma(t_f)=-v(t_f)\cos\gamma(t_f),$$

(18*c*)
$$\lambda_x(t_f)=-s_c[x(t_f)-X],$$

$$\lambda_h(t_f)=\frac{\dot{v}(t_f)}{v(t_f)}+\frac{\dot{\gamma}(t_f)+s_c[x(t_f)-X]}{\mathrm{tg}\gamma(t_f)};$$

2.3. Numerical solution

The two-point boundary value problem (7), (17), (18) is solved numerically. Eqs.(4) are integrated using the 4th order Runge – Kutta method until the first condition in (10) is fulfilled. Then, using eqs.(18) the end values of the costate variables are defined and eqs.(17) are integrated backwards until $t = t_0$. The consequent approximation for the control function is

(19)
$$\alpha^{N+1}(t)=\alpha^N(t)+k^N(t)\frac{d\mathrm{H}}{d\alpha^N(t)},$$

limited by the maximum allowed angle of attack α_s in eq.(8). For the gradient of the Hamiltonian an analytical form exists

(20)
$$\frac{d\mathrm{H}}{d\alpha^N(t)}=\mu c_L^\alpha H(h^N)v^N[-2v^N\lambda_v^N\alpha^N(t)+\lambda_\gamma^N].$$

The initial approximation $\alpha^0(t)$ for the first criterion is derived from the condition of maximum lift-to-drag ration on the trajectory, and for the particular model is

(21*a*)
$$\alpha^0(t)=\sqrt{c_{D0}(M)/c_L^\alpha}.$$

For the other two criteria the initial approximation was computed according to the formula

(21*b,c*)
$$\alpha^0(t)=k_1\alpha^*+k_2\left(\gamma+\mathrm{tg}^{-1}\frac{h}{X-x}\right)$$

and the limitation on the angle of attack (8). The parameters k_1,α^* and k_2 are explicitly defined in [5]. Here in parentheses is the direct pursuit control parameter, and α^* is determined from the condition for a straight flight

$$(22) \qquad \alpha^* = \frac{g\cos\gamma}{\mu c_L^\alpha H(h) v^2}.$$

The choice of $k^N(t)$ in (19) allows for incorporation of different optimization techniques and is a subject of a separate study. Effective numerical algorithms for the solution of the two boundary value problem in flight dynamics are studied in [7, 8].

3. Results and discussion

Figure 1 shows the initial approximation, the optimal control function, and the state variables on the optimal trajectory for the maximum range criterion. For v_0=450 m/s, g_0=30°, x_0=0, and h_0=10000 m the increase in the flight range is 42 percent.

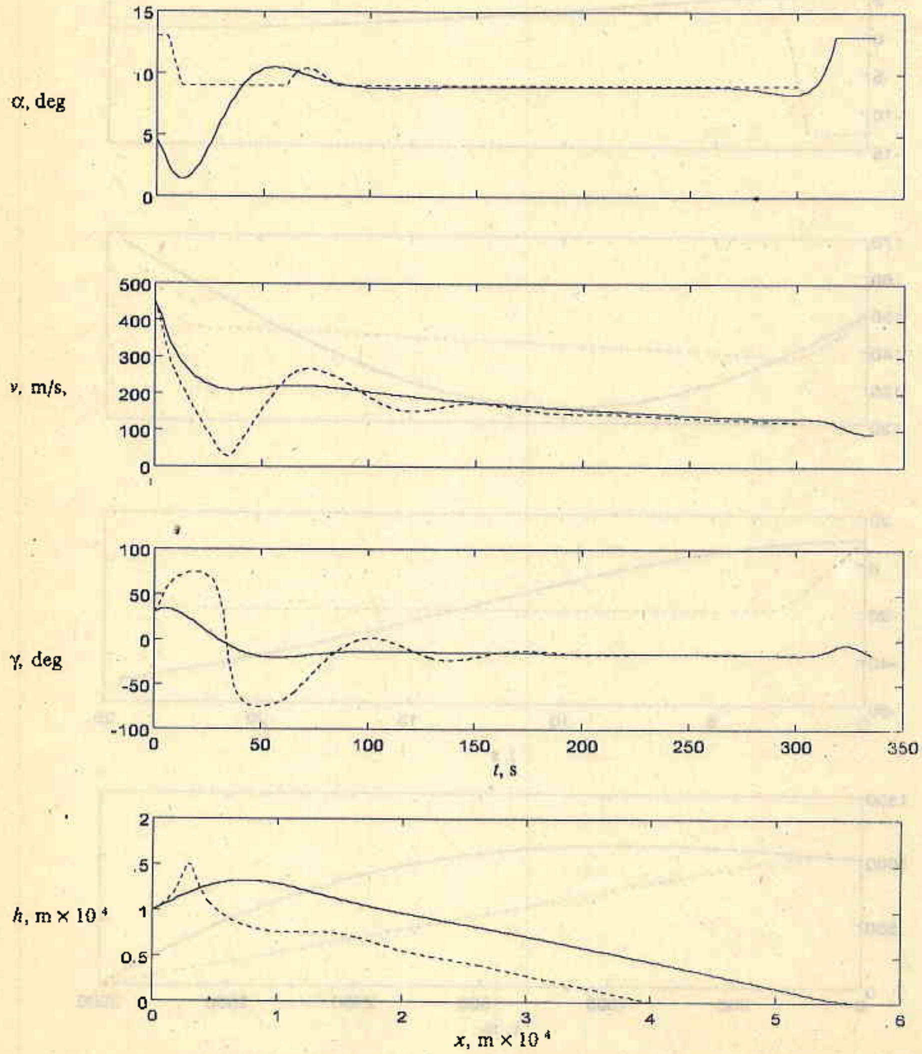

Fig.1. Optimal control and trajectory for the maximum range criterion
_ _ _ _ _ _ _ initial approximation, ________ optimal variables

The relevant variables for the maximum kinetic energy criterion are presented in fig. 2. For v_0=150 m/s, γ_0=10°, x_0=0, h_0=1000 m and X=3000 m the energy at the point of impact with the Earth surface is increased with 40 percent. For the same initial condition and X=5500 m the use of traditional control methods does not guarantee arrival at the required point of impact, while the optimal control does. If X=5900 m the required point of impact cannot be reached even with implementation of the optimal control function. The solution in such cases coincides with the optimal control for the maximum range criterion without, however, satisfying the second terminal constraint in (10*b*).

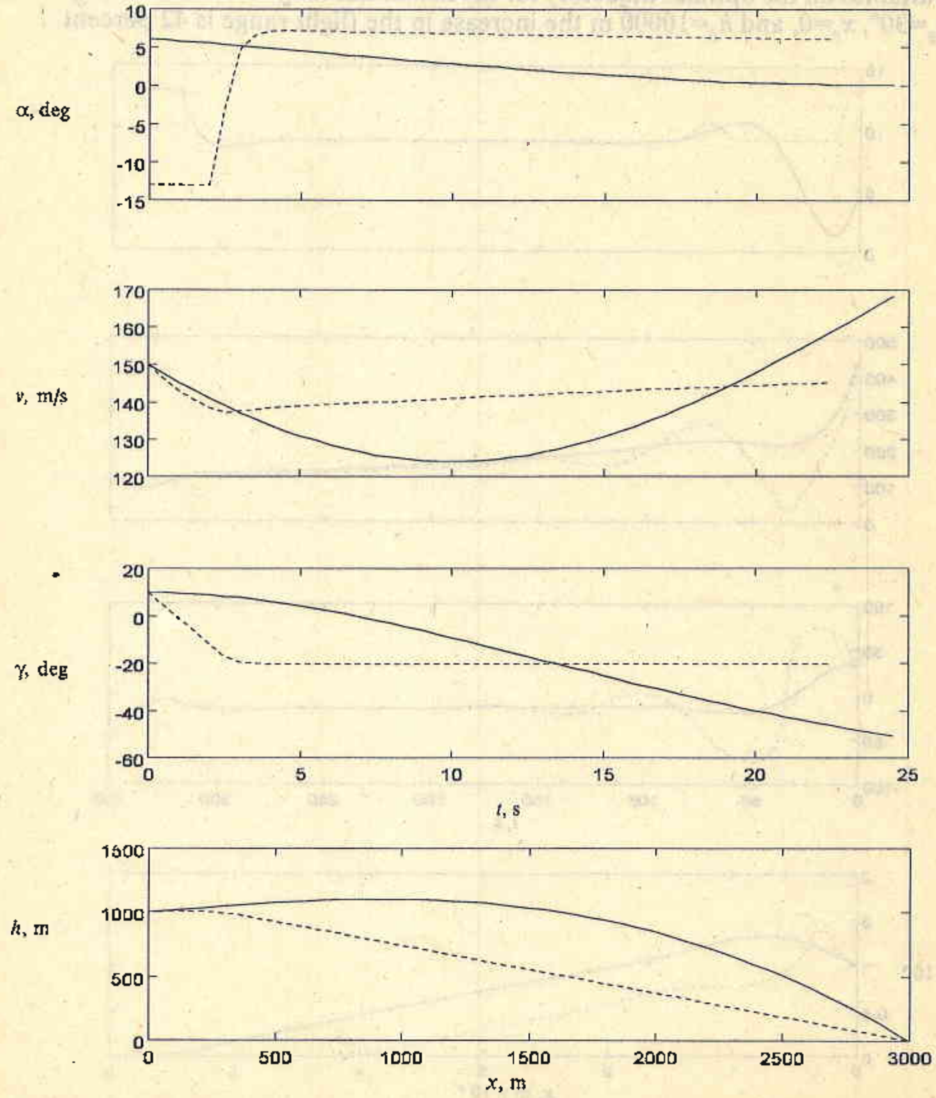

Fig.2. Optimal control and trajectory for the maximum kinetic energy at the point of impact criterion
_ _ _ _ _ _ _ initial approximation, ________ optimal variables

Figure 3 shows the initial approximation, the optimal control function, and the optimal trajectory variables minimizing criterion (11*c*). For initial conditions v_0=150 m/s, γ_0=10°, x_0=0, h_0=1000 m and X=3000 m the implementation of the optimal control would increase the depth of penetration with 117 percent. Furthermore, if the direct pursuit guidance is applied, the aircraft would meet the Earth surface at an angle of 20 degrees, which would result a significant probability for a ricochet.

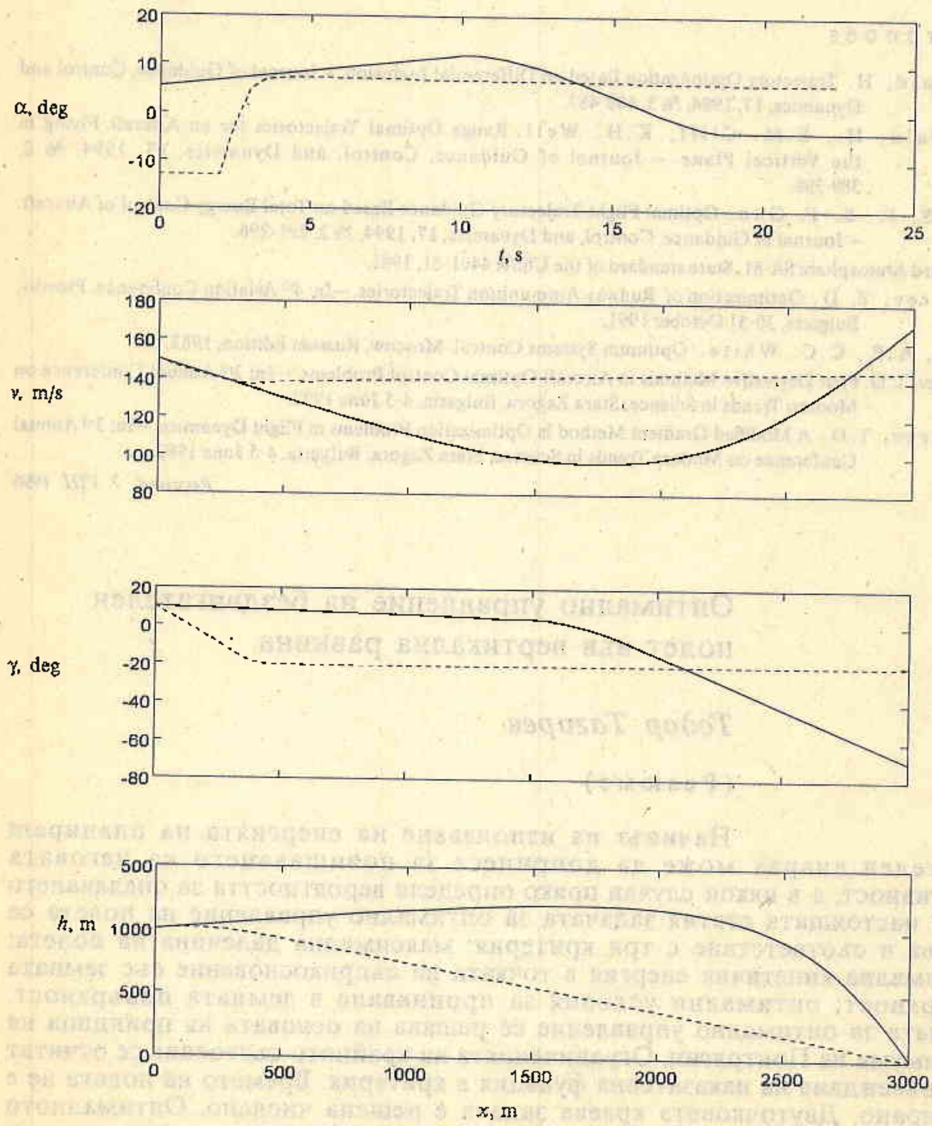

Fig.3. Control function and trajectory variables for optimal conditions for penetration at the point of impact with the Earth surface
_ _ _ _ _ _ initial approximation, ________ optimal variables

The implementation of the optimal control would increase not only the aircraft effectiveness, but also the opportunities for compensation for model parameter changes and other perturbations.

In conclusion, the optimal control problem of unpowered flight in vertical plane was solved via Pontryagin's minimum principle for a class of terminal criteria and constraints. The implementation of such control would increase the aircraft effectiveness, and in some cases, i.e., a flight of an aircraft after its engines have failed, may guarantee safe landing and survival of crew and passengers.

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Оптимално управление на бездвигателен полет във вертикална равнина

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(Резюме)

Начинът на използване на енергията на планиращ летателен апарат може да допринесе за повишаването на неговата ефективност, а в някои случаи пряко определя вероятността за оцеляването му. В настоящата статия задачата за оптимално управление на полета се решава в съответствие с три критерия: максимална далечина на полета; максимална кинетична енергия в точката на съприкосновение със земната повърхност; оптимални условия за проникване в земната повърхност. Задачата за оптимално управление се решава на основата на принципа на максимума на Понтрягин. Ограниченията на крайното състояние се отчитат чрез въвеждане на наказателна функция в критерия. Времето на полета не е фиксирано. Двуточковата краева задача е решена числено. Оптималното управление на планиращия полет гарантира ефективно използване на кинетичната и потенциалната енергия на летателния апарат. Реализацията му допълнително позволява компенсиране на промени в параметрите на модела и неотчетени външни смущения.