Българска академия на науките. Bulgarian Academy of Sciences
Аерокосмически изследвания в България. 10. Aerospace Research in Bulgaria
София.1993.Sofia

# Approximation of the out-of-focus intensity distribution for images having a Gaussian point source function

*Dimitr Dimitrov*

*Space Research Institute, Bulgarian Academy of Sciences*

## Introduction

It is well known that long exposure observations through a random turbulent medium by large aperture optical systems do not give diffraction limited spatial (angular) resolution. In that case the response function of the "telescope+atmosphere" system is not the Airy's diffraction pattern but an extended spot which size is one or even two orders of magnitude larger than the diameter of the Airy's central spot. In this paper we assume that the observed point source causes a Gaussian intensity distribution $g_0(r_0)$ of the light in the focal plane $(x_0, y_0)$ of the optical system

$$g_0\left(r_0=\sqrt{x_0^2+y_0^2}\right)=S_0 \exp\left(-r_0^2/2\sigma_0^2\right), \tag{1}$$

where $\sigma_0$ is a constant determining the size of the circular spot. $S_0=S_0(\sigma_0)$ is a normalization constant depending on the total energy flux of the image. We consider images on the principal optical axis and fully neglect the distortion effects like coma, astigmatism, etc. We also assume that $\sigma_0$ is much greater than the Airy's pattern and, consequently, the geometric optics approximation is a good approach.

## Definition of the problem

Let us denote by $(x', y')$ the input aperture plane and by $(x, y)$ the out-of-focus plane where the light sensitive detectors are placed. The origins of the coordinate systems lie on the principal optical axis and the corresponding coordinate axes are parallel and aligned in the same directions. We also denote by $f$ and $\Delta f$ the focal length and the distance between $(x_0, y_0)$ and $(x, y)$, respectively. Further we shall assume that the circular input aperture of the optical system has a central screening (as it is usual for the large telescopes). Consequently, the area of the input aperture is a ring with inner radius $r_1'$ and outer radius $r_2'$. Let us denote by $h(x, y)$ the point-spread function of this out-of-focus optical system, i. e., this is the intensity distribution in the out-of-focus plane $(x, y)$ when the system is illuminated by light rays parallel to the principal axis. According to the accepted geometrical optics approach, the illuminated area is a ring with inner radius $r_1$ and outer radius $r_2$ $(r_2'/f=r_2/\Delta f)$. Obviously, the screened part of the input (output) aperture is characterized by the relations $r_1'/r_2'=r_1/r_2=\mathrm{E}=\mathrm{const}<1$. Therefore, within a normalizing multiple, $h(x, y)$ is given by

$$(2)\qquad \tilde{h}(r=\sqrt{x^2+y^2})=\begin{cases}1, & \text{if } r_1\le\sqrt{x^2+y^2}\le r_2;\\ 0 & \text{in the other cases.}\end{cases}$$

We shall investigate the distortions caused by the out-of-focus registration of the intensity distribution $g(x, y)$ conditioned by a point source observed through a turbulent medium. Supposing that the principle of the linear superposition is fulfilled, we may write

$$(3)\qquad g(r)\equiv g(x,\ y=0)=\int\int_{-\infty}^{\infty} g_0(x-\xi,\ -\eta)\,h(\xi,\eta)\,d\xi\,d\eta,$$

where we have taken into account the circular symmetry of the image. It would be stressed that the coordinates $x$, $y$, $\xi$ and $\eta$ are referred to the out-of-focus plane, while the coordinates $x_0$ and $y_0$ are referred to the focal plane.

The observed intensity distribution $g(x, y)$ may be fitted by a quasi-Gaussian function

$$(4)\qquad g(r)\equiv g(x,\ y=0)=S\exp\left[-x^{2n(x)}/B\right];\quad B=(2\sigma^2)^{n(x)},$$

and, generally speaking, we expect that for this approximation the power $n$ is not a constant and will depend on $r$ (or $x$, because we investigate the radial distribution in the direction $y=0$). It is not difficult to obtain an analytical expression for the global behaviour of the power $n(x)$ if we assume that this quantity does not vary too fast into the interval $(x-\Delta x, x+\Delta x)$, where $\Delta x\ll\sigma$. Taking the natural logarithms from the left and right hand sides of the equation (4) and differentiating with respect to $x$, after some trivial algebra we obtain [1]

$$(5)\qquad n(x)=0{,}5\left[1+x\left(\frac{g''(x)}{g'(x)}-\frac{g'(x)}{g(x)}\right)\right],$$

where the prime's denote differentiating with respect to $x$. By means of the above approximate expression we shall evaluate the global (with respect to the size of the image) changes of the power $n$ which describes the slope of the intensity distribution $g(x)$.

Hereafter in this paper we shall consider $\sigma_0$ as a strictly positive quantity $(\sigma_0>0)$ setting also $S_0=1$ for a fixed value of $\sigma_0$. The case $\sigma_0=0$ has a trivial interpretation: the point-spread function of the turbulent medium is a $\delta$-function and (in the geometrical optics approach) the out-of-focus intensity distribution $g(x, y)$ is a circular ring with a constant nonzero value. Out of the ring the intensity is equal to zero. It is useful to introduce polar coordinates $\rho$ and $\theta$

(6) $$\xi=\rho\cos\theta; \quad \eta=\rho\sin\theta.$$

Substituting (1), (2) and (6) into (3), we obtain

(7) $$g(x, y=0)=g(x)=\int_0^{\infty}\rho h(\rho)\int_{-\pi/2}^{3\pi/2}\exp\left[-\frac{x^2+\rho^2-2x\rho\cos\theta}{2\sigma_0^2}\right]d\theta\, d\rho$$

$$=\int_{r_1}^{r_2}\rho A(x\rho/\sigma_0^2)\exp[-(x^2+\rho^2)/2\sigma_0^2]\,d\rho,$$

where we have denoted by $A(x\rho/\sigma_0^2)$ the expression

(8) $$A(x\rho/\sigma_0^2)=\int_{-\pi/2}^{3\pi/2}\exp(x\rho\cos\theta/\sigma_0^2)\,d\theta=2\pi J_0(-ix\rho/\sigma_0^2).$$

In the above equality $J_0$ is the first kind Bessel function of order zero [2,3]. Further we shall use the development of the Bessel function $J_0$ in powers of its argument [2,3]. We find that

(9) $$A(x\rho/\sigma_0^2)=2\pi\sum_{k=0}^{\infty}(k!)^{-2}(x\rho/2\sigma_0^2)^{2k}.$$

As can be seen from the numerical values of the first eight terms ($k=0, 1, \ldots, 7$), the above expression shows a fast convergence. In practice, it is enough to restrict (9) to a partial sum with a maximal value of $k$ equal to $6\div 7$. The obtained accuracy is sufficient for the considered range of values of $(x\rho/\sigma_0^2)$. According to (9), the radial intensity distribution $g(x)$ (7) may be rewritten in the following way

(10) $$(2\pi)^{-1}g(x)=\sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)^2\int_{r_1}^{r_2}\rho\left(\frac{x\rho}{2\sigma_0^2}\right)^{2k}\exp\left[-\frac{x^2+\rho^2}{2\sigma_0^2}\right]d\rho,$$

or

(11) $$(2\pi)^{-1}g(x)=B_0(x)+\sum_{k=1}^{\infty}\left[\left(\frac{1}{k!}\right)^2\left(\frac{x}{2\sigma_0^2}\right)^{2k}\right]B_k(x),$$

where we have set

(12) $$B_0(x)=\sigma_0^2\left\{\exp\left[-\frac{x^2+r_1^2}{2\sigma_0^2}\right]-\exp\left[-\frac{x^2+r_2^2}{2\sigma_0^2}\right]\right\}$$

and

$$(13)\quad B_k(x)=\int_{r_1}^{r_2}\rho^{2k+1}\exp[-(x^2+\rho^2)/2\sigma_0^2]\,d\rho=\sigma_0^2\{r_1^{2k}\exp[-(x^2+r_1^2)/2\sigma_0^2]$$

$$-r_2^{2k}\exp[-(x^2+r_2^2)/2\sigma_0^2]\}+2k\sigma_0^2 B_{k-1}(x);\ (k=1,2,\ldots).$$

Differentiating (11) with respect to $x$ two times, we obtain, correspondingly

$$(14)\qquad (2\pi)^{-1}g'(x)=B_0'(x)+\sum_{k=1}^{\infty}\left[\frac{2k}{(k!)^2}\left(\frac{1}{2\sigma_0^2}\right)^{2k}\right]x^{2k-1}B_k(x)$$

$$+\sum_{k=1}^{\infty}\left[\frac{1}{(k!)^2}\left(\frac{x}{2\sigma_0^2}\right)^{2k}\right]B_k'(x);$$

$$(15)\qquad (2\pi)^{-1}g''(x)=B_0''(x)+\sum_{k=1}^{\infty}\left[\frac{2k(2k-1)}{(k!)^2}\left(\frac{1}{2\sigma_0^2}\right)^{2k}\right]x^{2k-2}B_k(x)$$

$$+2\sum_{k=1}^{\infty}\left[\frac{2k}{(k!)^2}\left(\frac{1}{2\sigma_0^2}\right)^{2k}\right]x^{2k-1}B_k'(x)+\sum_{k=1}^{\infty}\left[\frac{1}{(k!)^2}\left(\frac{x}{2\sigma_0^2}\right)^{2k}\right]B_k''(x).$$

In the above expressions the coefficients $B_0'(x)$, $B_0''(x)$, $B_k'(x)$ and $B_k''(x)$ may easily be computed analytically from (12) and (13) by differentiating them one or two times. For brevity, we shall not write in an explicite manner these derivatives.

Having the estimations (11), (14) and (15), we are able to estimate also the ratios $g''(x)/g'(x)$ and $g'(x)/g(x)$. According to (5), the out-of-focus power $n(x)$ is evaluated through the infinite series which may be truncated at some value of the summation index $k$.

## Numerical evaluation of the intensity distribution

With a view to evaluate the intensity distribution $g(x)$ and power $n(x)$ of the out-of-focus point sources, we have performed numerical evaluations of the series entering into the expressions (11), (14) and (15) for different meanings of the argument $x/\sigma_0$ (i. e., all linear parameters in the plane $(x, y)$ are measured in units of $\sigma_0=1$). It is sufficient to truncate the infinite series at $k=6$. Disregarding of the terms with $k\geq 7$ leads to noticeable errors only in the case of very strong out-of-focus distortions if $x/\sigma_0\gtrsim 3{,}4\div 3{,}6$. The degree of the out-of-focus distortions of the images may be characterized by the inner radius $r_1$ instead of the distance $\Delta f$ between the planes $(x_0, y_0)$ and $(x, y)$. We have considered three cases of distortions, which we shall for convenience denote as "slight", "moderate" and "strong":

(*i*) "slight" out-of-focus distortions: $r_1/\sigma_0=0{,}50$;

(*ii*) "moderate" out-of-focus distortions: $r_1/\sigma_0=0{,}86$;

(*iii*) "strong" out-of-focus distortions: $r_1/\sigma_0=1{,}00$.

In this case a decrease of the intensity in the central region (relative to the intensity for little larger radii) is observed. This reduction is caused by the central screening of the input aperture ($r'_1>0$). Obviously, in the case (*iii*) the approximation (4) is not more valid when we try to describe the total (global) intensity distribution $g(x)$. But we still may try to use the analytical expression (5) in order to evaluate the power $n(x)$, describing the slope of the function $g(x)$.

Of course, if $r_1=0$, the out-of-focus distortions absent. Here we remember that the screening parameter $E=r'_1/r'_2$ has a constant value. For a concreteness, we have adopted E=0,43. Undoubtedly, if we use other values of E, the results will qualitatively be the same. Since the parameter $S_0(\sigma_0)$ is not specified in this paper, the intensities $g(x)$ are measured in arbitrary units. We have normalized the out-of-focus intensities in such a way, that the maximal values of $(1/2\pi)g(x)$ are equal to 100% (Fig. 1). In this figure the $e^{-1}=$ 36,8% intensity level is indicated by a straight horizontal line. The numbers above the arrows designate the radii of the smoothed out-of-focus point source images. It is evident that the intensity drop in the center of the image ($x=0$) for "strong" out-of-focus distortions appears when the size of the image is enlarged about two times in comparison with the precise focused image ($r_1/\sigma_0=0$). Of course, such a large increasing of the sizes of the images cannot remain unnoticed by the observer even if the images are visually focused.

The out-of-focus values of the power $n(x)$, computed according to (5), are presented in Fig. 2. For a "slight" distortions $n(x)$ is close to unity (because the addopted initial distribution $g_0(x)$ is a Gaussian one) and only when the intensity $g(x)$ considerably drops to a low level, $n(x)$ monotonically increases by $\sim$10 per cent. This behaviour of the power $n(x)$ may be considered as a typical for enlargements of the images $\Delta\sigma$ which do not exceed$\sim(1/4)\sigma_0$. In the case of "moderate" distortions $n(x)$ strongly deviates from the initial power $n_0=1$, but in this situation the increasing of the image radius is also large ($\Delta\sigma\sim(0,7\div0,8)\sigma_0$) and cannot remain unnoticed. For "strong" out-of-focus distortions our description of the intensity distribution $g(x)$ by the approximation (4) obviously fails. The power $n(x)$, evaluated by means of (5), is not a continuous function of its argument $x$ and decomposes into separate branches with very different (varying) values of $n(x)$ (Fig. 2). As mentioned above,

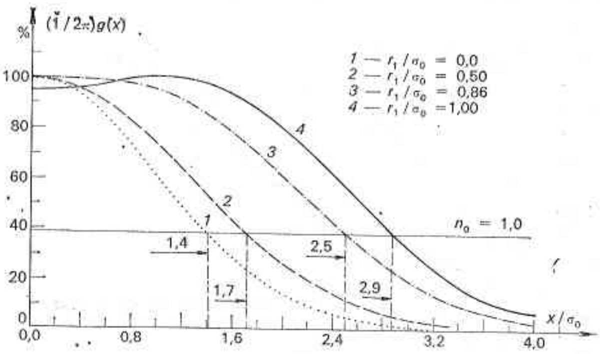


Fig. 1. Out-of-focus intensity distribution $(1/2\pi)g(x)$ for the Gaussian case of precise focused images. The maximal values are normalized to 100% and the numbers above the arrows indicate the size of the images at 36, 8% level

*1* — exact focused image; *2* — "slight"; *3* — "moderate"; *4* — "strong" out-of-focus distorted images

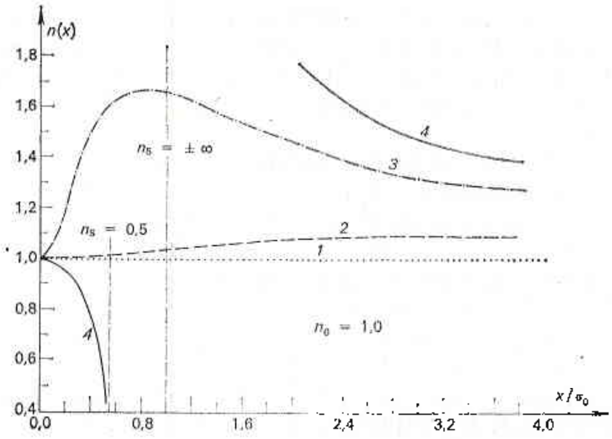


Fig. 2. Out-of-focus power $n(x)$ for the Gaussian case of precise focused images. Vertical thin dashed lines indicate values of $x$ where $n_s=0,5$ and $n_s=\pm\infty$, respectively (the letter "s" denotes "strong" distortions). The other notations are the same as in Fig. 1

this case may be excluded as an extraordinary situation of out-of-focus images and, consequently, there is not reason to reject the approximations (4) and (5) at all.

In practice, nearly constant values of the power $n(x)$ may be determined from the slope of the function $S(x)$ (if $g(x)$, $g'(x)$ and $g''(x)$ are already known from observations)

$$(16) \qquad S(x)=\left[\frac{g''(x)}{g'(x)}-\frac{g'(x)}{g(x)}\right]^{-1}=\frac{1}{2n(x)-1}x,$$

for $n(x)\approx$constant (locally). From Fig. 3 it is evident that if the out-of-focus distortions are not very large, the mean value of the power $n(x)$ may be determined effectively by means of a linear approximation of $S(x)$ (i. e., $S(x)$ is a suitable tool for measurement of these distortions from the experimental

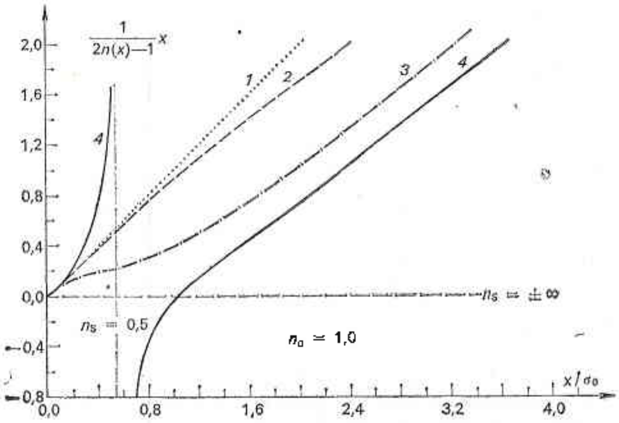


Fig. 3. Function $S(x)=[2n(x)-1]^{-1}x$ for the Gaussian case of precise focused images. Notations are the same as in Fig. 2

data). Excluding the "strong" out-of-focus distortions, we conclude that in the case of a Gaussian point-spread function of the turbulent medium, the values of the power $n(x)$ are always increased with an amount which is very sensitive to the distance between the focal plane $(x_0, y_0)$ and the plane $(x, y)$, where the intensity $g(x, y)$ is measured.

## Conclusions

We have derived analytical expressions describing the out-of-focus intensity distribution arizing from a point source observed through a turbulent medium. An optical system with central screening of the input aperture is considered. All computations use the geometrical optics approach. If out-of-focus distortions are not too large, it is possible to approximate this intensity distribution by a quasi-Gaussian curve with a power $n(r)$ depending on the distance $r$ from the center of the image. If such images are scanned, for example, by an infinitely long slit diafragm with a finite width, the measured intensity distribution will be additionally enlarged, but this smearing out is about $(1 \div 2)$ per cent [4]. As can be seen from Fig. 1, the enlargement of the image sizes due to the out-of-focus distortions may be much more pronounced than the enlargement due to the scanning. Consequently, measuring the behaviour of the power $n(r)$ may give a valuable information about the degree of the out-of-focus distortions of the observed objects. For example, in the case of astronomical observations by ground-based telescopes, it is sufficient to investigate turbulent star images (as point sources). It may be shown [1, 5] that for precisely focused turbulent star images the power $n_0$ is less than unity (despite of that is not a constant). If the measured intensity distribution $g(r)$ can be approximated by a quasi-Gaussian curve (4) with a power $n(r)$ exceeding appreciably unity, this may be an indication that the images are not precisely focused. Such a method for evaluation of the out-of-focus effect is possible to be combined with the usual procedure of minimizing the sizes of the point source images.

Aknowledgements. The autor thanks Drs. Yu. V. Alexandrov and V. N. Dudinov from the K'harkov State University (Ukraina) for helpful discussions and comments on the manuscript.

## References

1. Dimitrov, D. V. Investigation of the photoelectric profiles deviations of the stellar images from the Gaussian distribution. — Vestnik K'harkov univ., ser. 15, 1980, No 204, p. 44.
2. Jahnke, E., F. Emde, F. Lösch. Special functions. Formulas, graphs and tables. M., Nauka, 1977, p. 176.
3. Abramowitz, M., A. Stegun. Handbook of mathematical functions with formulas, graphs and mathematical tables. M., Nauka, 1979, p. 177.
4. Dimitrov, D. V. Scanning of stellar images by a slit diafragm and estimation of their parameters. I. Approximation of turbulent disks by a Gaussian distribution. — Astrophysical investigations, 7 (in print).
5. Dimitrov, D. V. Two-parametric intensity distribution in the turbulent stellar disks. — Vestnik K'harkov univ., ser. 15, 1980, No 204, p. 40.

*Received 2. XI. 1992*

# Апроксимация на извънфокалното разпределение на осветеността при изображения, имащи гаусова функция на импулсния отклик

*Димитър Димитров*

(Резюме)

Прието е функцията на импулсния отклик при наблюдения през турбулентна среда да бъде гаусова крива $g_0(r) \sim \exp\left(-\frac{r^2}{2\sigma_0^2}\right)$. Ако измерванията на интензивността не са извършени във фокалната равнина на телескопа, то изопаченото разпределение на осветеността $g(r)$ може да бъде апроксимирано с квазигаусова крива $g(r) \sim \exp(-r^{2n}/B)$, където степенният показател $n$ зависи от $r$. Показано е, че това приближение е подходящо описание на $g(r)$, ако извънфокалните изопачавания не са много големи. Извършени са детайлни аналитични и числени изчисления с оглед да се оцени $n(r)$ и неговото отличие от единица. Всички оценки са извършени в приближението на геометричната оптика.