BULGARIAN ACADEMY OF SCIENCES

SPACE RESEARCH IN BULGARIA, 1

Sofia · 1978

On the Differential Rotation and the Figure of Celestial Bodies*

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1. Introduction

The differential rotation of the celestial bodies is a topical problem. Attempts to explain this phenomenon have heen made by Kalitzin [1], Clement [2], Rubashev [3], Menzel [4], Fessenkov and others. Lichkov [5] considers the rotation of the Earth's envelopes (nucleus, mantle, litosphere, hydrosphere and atmosphere) with different angular velocities. The differential rotation is clearly observed in the Sun, Jupiter, Saturn and in our Galaxy.

The purpose of this work is to provide a reasonable solution to this important problem, taking into consideration the fact that *gravitation* is the principal factor in this phenomenon.

2. Description of Model

We first consider the celestial body in its earliest stage of evolution, representing it by the following idealized model. We assume that the body consists of elementary layers with equal eccentricity. The body may be homogeneous or with increasing density toward the centre, according to any law. The viscosity is neglected: we assume that at the extremely high temperature of the young celestial body the viscosity is equal to zero. The figure of this ideally elastic body is determined by the action only of gravitation and of the centrifugal force, and it is assumed to be an oblate ellipsoid of rotation. Our model is very near to the structure of the stars from the early spectral classes and that of neutron stars, described by Shklovskiy [6], whose superfluid matter, deprived of viscosity, is of an ellipsoidal equilibrium configuration.

*For open discussion

3. Methods and Results

1. We use Newton's condition of equilibrium:

$$P - E_0 = F_0, \ [7]$$

in its generalized form

$$P - E_\varphi = F_\varphi \cos \varphi, \tag{1}$$

or

$$E_\varphi + F_\varphi \cos \varphi = P, \tag{2}$$

(see Fig. 1)

where P is the weight of the polar column with length b (the polar semi-axis), E_0 is the weight of the equatorial column with length a (the equatorial semi-axis), F_0 is the sum of the centrifugal forces of the particles (elementary layers) of the equatorial column, E_φ is the weight of the column with latitude φ and with length c, equal to the radius-vector of the ellip-

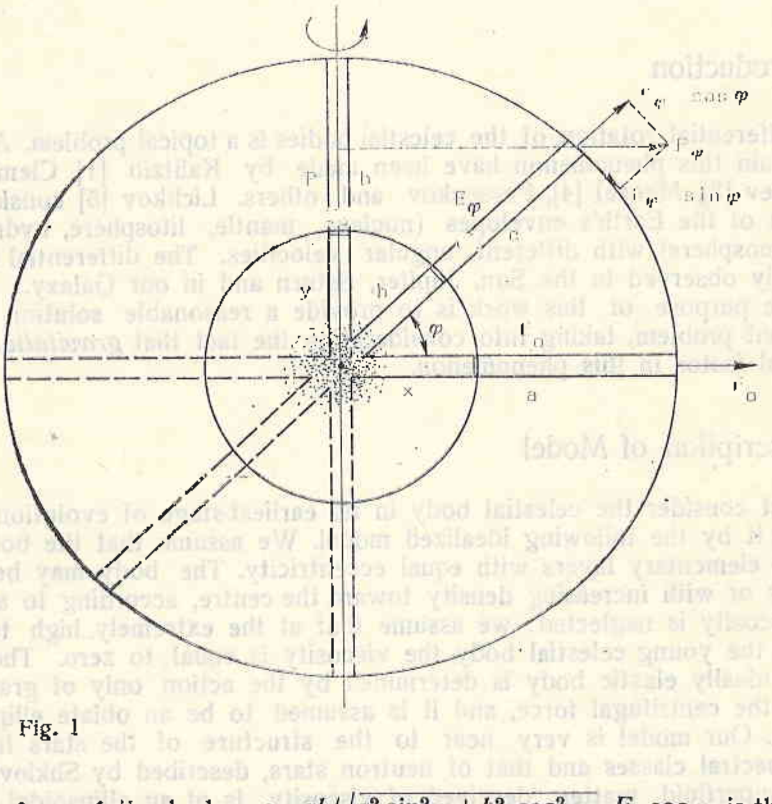

Fig. 1

soidal surface of the body: $c = ab/\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}$, $F_\varphi \cos \varphi$ is the sum of the radial components of the centrifugal forces of the elementary layers of the column E_φ. The cross-section of both columns connecting the centre of the celestial body with the pole and with any point on the surface with latitude $0° \leq \varphi \leq 90°$, is equal to unity. We must add that P, E_0, F_0, E_φ and $F_\varphi \cos\varphi$ are considered as *scalar* quantities.

We take the law of Legendre-Roche [8] for the density alteration in its common form:

$\bar{\varrho}=\varrho[1-\alpha(x/R)^z]$ — for alteration of the average density of the compound ellipsoids of the body;

$\varrho_s=\varrho[1-\beta(x/R)^z]$ — for alteration of the surface density of the compound ellipsoids, or of the compound elementary layers (envelopes) of the body, where ϱ is the density in the centre of the body.

At $z=0$ and $\alpha=\beta$ the body is homogeneous. Giving different values to the constants z, α and β we could represent an infinite number of celestial bodies, from homogeneous ones to such with strongly increasing density to the centre, i. e. with an inner structure similar to Roche's model.

We express the condition of equilibrium (2) of the celestial body by the following integral equation:

$$\int_0^c G\frac{(4/3)\pi x^2 y\varrho[1-\alpha(h/c)^z]\varrho[1-\beta(h/c)^z]}{h^2}dh+\int_0^c \omega_h^2 h\cos^2\varphi\varrho[1-\beta(h/c)^z]\,dh$$
$$=\int_0^b G\frac{(4/3)\pi x^2 y\varrho[1-\alpha(y/b)^z]\varrho[1-\beta(y/b)^z]}{y^2}dy, \tag{3}$$

where x and y are the semi-axes of the attracting compound ellipsoid, h is the distance from the centre of the body to the surface of the attracting ellipsoid at latitude φ; $h=xy/\sqrt{x^2\sin^2\varphi+y^2\cos^2\varphi}$, ω_h is the angular velocity of the elementary zonal layer at a distance h from the centre, with latitude φ. The expression

$$(4/3)\pi x^2 y\varrho[1-\alpha(h/c)^z]=(4/3)\pi x^2 y\varrho[1-\alpha(y/b)^z],$$

in equation (3), is the mass of the attracting ellipsoid and the expression $\varrho[1-\beta(h/c)^z]dh=\varrho[1-\beta(y/b)^z]\,dy$ is the mass of the elementary layer in columns c and b. The equalities $h/c=y/b=x/a$ in the above expressions follow from the equality of the eccentricity of the elementary ellipsoidal layers (envelopes) of the body: $\sqrt{(a^2-b^2)/a^2}=\sqrt{(x^2-y^2)/x^2}$.

In equation (3) the attraction of the elementary layer in the columns is expressed simply by Newton's law of gravitation, but with sufficient exactness, as the attracting mass of rapidly rotating celestial bodies is assumed to be concentrated toward the centre, and the form of slowly rotating celestial bodies is almost a spherical one. This is confirmed by the accuracy of the final results.

The following proportion obviously exists with the homogeneous celestial body:

$$\frac{\omega_0^2 a}{\omega_0^2 x}=\frac{G(4/3)\pi a^2 b\varrho(1-\alpha)/a^2}{G(4/3)\pi x^2 y\varrho(1-\alpha)/x^2}, \tag{4}$$

which expresses the equality of the ratios of the centrifugal and gravitational accelerations at the surface and at any distance to the centre of the body. Here the centrifugal force and the attraction are changing linearly to the centre, where they are equal to zero. Proportion (4) expresses the inner condition of equilibrium of the homogeneous celestial body.

For a celestial body presented by our idealized model with increasing density to the centre, according to Legendre-Roche's law, however, proportion (4) should take the form

$$\frac{\omega_0^2 a}{\omega_x^2 x}=\frac{G(4/3)\pi a^2 b\varrho(1-\alpha)/a^2}{G(4/3)\pi x^2 y\varrho[1-\alpha(x/a)]^2/x^2}, \tag{5}$$

as for the inner equilibrium of such a body the centrifugal force and the attraction to the centre must also change with the distance to an equal degree. Here $\omega_x > \omega_0$.

Proportion (4) appears as a particular case of proportion (5). Actually, at $z=0$ and $y=xb/a$ we have $\omega_x=\omega_0$.

At latitude φ proportion (5) has the following form:

$$\frac{\omega_\varphi^2\, c\cos\varphi}{\omega_h^2\, h\cos\varphi}=\frac{G(4/3)\pi a^2 b\,\varrho(1-\alpha)/c^2}{G(4/3)\pi x^2 y\,\varrho[1-\alpha(h/c)^2]h^2}, \tag{6}$$

where ω_φ is the angular velocity of the elementary zonal layer on the surface of the body, with latitude φ.

From the equalities $(a^2-b^2)/a^2=(x^2-y^2)/x^2$, $c=ab/\sqrt{a^2\sin^2\varphi+b_2\cos^2\varphi}$, $h=xy/\sqrt{x^2\sin^2\varphi+y^2\cos^2\varphi}$ and proportion (6) we have: $x^2=y^2a^2/b^2$, $h^2=c^2y^2/b^2$, $y=hb/c$ and $\omega_h^2=\omega_\varphi^2(c^z-ah^z)/(c^z-ac^z)$.

Substituting the above values of x^2, h^2, y and ω_h^2 in equation (3), followed by simplification and integration, we obtain

$$\frac{G(4/3)\pi a^2 b\varrho(1-\alpha)}{c}+\omega_\varphi^2 c^2\cos^2\varphi=\frac{G(4/3)\pi a^2 b\varrho(1-\alpha)}{b}, \tag{7}$$

where $\varrho(1-\alpha)$ is the average density of the body.

Introducing the mass of the celestial body, we obtain

$$GM/c+\omega_\varphi^2 c^2\cos^2\varphi=GM/b, \tag{8}$$

from where

$$t_\varphi=2\pi c\cos\varphi\sqrt{\frac{cb}{GM(c-b)}}, \tag{9}$$

where $t_\varphi=2\pi/\omega_\varphi$ is the rotational period of the elementary zonal layer at a distance c from the centre, at latitude φ.

2. The same result (9) is also obtained from the equation

$$V_\varphi+U_\varphi=V_p, \tag{10}$$

which at $V_p=$const describes an equipotential surface. Here V_φ is the inner gravitational potential, in our model, of a point on the surface with latitude φ, i. e. the work for the transport of unit mass from the centre to the surface with latitude φ which is different from $\int_c^\infty \frac{GM}{h^2}\,dh$, V_p is the inner gravitational potential of the pole which is also different from $\int_b^\infty \frac{GM}{y^2}\,dy$, and U_φ is the potential of the centrifugal force at latitude φ, at the corresponding acceleration of the rotation to the centre.

Equation (10) which also expresses the condition of equilibrium of the celestial body, in our model, can be presented in the following integral form:

$$\int_0^c \frac{G(4/3)\pi x^2 y\varrho\,[1-a(h/c)^2]}{h^2}\,dh + \int_0^c \omega_h^2 h \cos^2\varphi\,dh = \int_0^b \frac{G(4/3)\pi x^2 y\varrho[1-a(y/b)^2]}{y^2}\,dy. \tag{11}$$

From Equation (11) we obtain, in a similar way, formula (9).

Equations (2) and (10) are equivalent. Newton's concept of "weight" of the column, expressed as $E_\varphi + F_\varphi \cos\varphi$, corresponds to the inner potential at the same latitude, namely: $V_\varphi + U_\varphi$!

3. In our model, Newton's condition of equilibrium (2) can be expressed as follows:

$$E_\varphi/n + F_\varphi \cos\varphi/n = P/n, \tag{2'}$$

where $n \geq 1$. At $n \to \infty$ we can write

$$\frac{GM(c/n)\varrho_s}{c^2} + \frac{4\pi^2 c \cos^2\varphi (c/n)\varrho_s}{t_\varphi^2} = \frac{GM(b/n)\varrho_s}{b^2}, \tag{2''}$$

or

$$GM/c + 4\pi^2 c^2 \cos^2\varphi / t_\varphi^2 = GM/b, \tag{2'''}$$

from where formula (9) is directly obtained.

Formula (9) describes the differential rotation of the celestial bodies in our model.

4. Particular Cases of the Law (9)

At $\varphi = 0°$ formula (9) takes the form

$$t_0 = 2\pi a \sqrt{\frac{ab}{GM(a-b)}}. \tag{12}$$

Formula (12) can also be written as:

$$(a-b)/b = F/E,$$

where $F = \omega_0^2 a$ and $E = GM/a^2$, i. e. the *second* flattening of the celestial body is equal to the ratio of the centrifugal and gravitational accelerations, measured at the equator.

Similar results had been obtained by.

Newton: $(a-b)/a=5F/4E$ and

Huygens: $(a-b)/a=F/2E$ [7].

It is very important to note here that formula (12) can be obtained directly from the proportion

$$GM/r^2-GM/a^2=k\omega_0^2 a,$$

at $k=1$ and $r^2=ab$, where r is the radius of the ideally elastic celestial body at $\omega=0$. The relation $r=\sqrt{ab}$ is a consequence from Hooke's law and can be demonstrated experimentally, by axial rotation of an elastic and isotropic sphere.

At $(a-b)/a=1/2$ formula (12) takes the form

$$t_0=2\pi a\sqrt{\frac{a}{GM}}. \tag{14}$$

Formula (14) is the mathematical expression of Kepler's third law, for circular orbits. Actually, the equatorial particles of some stars from the early spectral classes B, A and F, which have very rapid axial rotation and in which the centrifugal force at the equator is almost equal to the attraction, are rotating as small planets, according to (14). The flattening $(a-b)/a$ of these stars must be almost equal to 1/2.

5. Verification of the Results

At the following values of the mass and semi-axes of the Earth (considered ideally elastic, as a whole): $M=5.98\times10^{27}$ g, $a=6378.245\times10^5$ cm, and $b=6356.863\times10^5$ cm (the ellipsoid of Krassovsky), formula (12) gives the following value for the rotational period of the Earth (more exactly for the period of the earth-crust): $t=87384$ s $=24.27$ h, with a relative error of about 1.4 per cent. At the same values of M, a and b of the Earth, Newton's theorem gives $t=27.18$ h, and that of Huygens gives $t=17.19$ h.

Formula (12) gives the rotational periods of the other planets as well, at the correct values of their masses and semi-axes. It appears to be the most exact, compared with the similar results of Newton, Huygens, Claitaut [7] and that of Radau — Darwin [9] which is quite unfit for the planets of the Jupiter type.

As the density of the Earth increases toward the centre, where the temperature is higher and the viscosity lower, we should have, according to (12) an acceleration of the rotation of its inner layers. This is in agreement with the conclusions of Munk and Macdonald [10] and of Lichkov [5].

At the following values for the mass, the equatorial radius and the rotational period at the equator of the Sun, namely: $M=1.99\times10^{33}$ g, $a=695500\times10^5$ cm, and $t_0=25\times86164$ s, formula (12) gives the following value for the polar radius of the Sun: $b=695485\times10^5$ cm. At these values for a and b we obtain $(a-b)/a=2.15\times10^{-5}$. It is interesfing to note that our theoretically determined value for the oblateness of the Sun is of the same order as that of Dicke: 5×10^{-5} [11], found experimentally.

At the following value for the density in the centre of the Sun: $\varrho = 120$ g/cm^3 [4], formula (12) gives a significant acceleration of the rotation of the inner layers of the Sun. This is in agreement with the conclusions of Dicke [11], Roxburgh [12], Fessenkov and other scientists that the inner layers of the Sun rotate more rapidly than the outer layers.

The mechanical energy of the differentially rotating layers of the celestial body is no doubt turning, at the friction between the layers, into thermal and other kinds of energy. This enormous source of energy must be taken into consideration in the solution of some astrophysical and planetary problems. For example, the total thermal flux from Jupiter is 2.5 times that which the planet receives from the Sun (from measurements made by Pioneer 10). This enigmatic phenomenon can be explained by the differential rotation of Jupiter, according to (9) and (15).

At the above values for the mass and semi-axes of the Sun, represented by our ideal model, formula (9) gives the following results for the rotational periods, in days, of the zonal layers of the photosphere of the Sun:

Table 1

φ	0°	30°	50°	70°	80°	85°
t_φ	25.0	24.9	24.8	24.5	22.6	17.5

As can be seen from the above Table, for a celestial body whose figure is an ideal ellipsoid, with viscosity equal to zero, formula (9) gives a "polar acceleration" of rotation. Actually, analyzing the line profiles of stars on the upper main sequence, Stoeckly [2] has concluded that these stars rotate more rapidly at the pole than at the equator. This is in agreement with our theoretical results, as the viscosity of these stars, which have a very high temperature, is negligible and their figure is almost an ideal ellipsoid of rotation.

On the other hand, as also noted by Stoeckly, stars on the lower main sequence, such as the Sun, possess an "equatorial acceleration". This phenomenon, observed also at the Sun, Jupiter and Saturn [1, 13, 4] could be explained by the action (influence) of the viscosity, a factor which should be taken here into account. The influence of the viscosity is, no doubt, reflected in the change of the periods of rotation of the zonal layers, in the change of the equipotential surface of the body and, consequently, in the change of the figure of celestial body.

Taking into account the integral influence of the viscosity, formula (9) should be written in the following form, for the older celestial bodies:

$$t_\varphi = 2\pi c_x \cos\varphi \sqrt{\frac{c_x b}{GM(c_x - b)}}, \tag{15}$$

where c_x is the radius-vector of the deformed figure of the celestial body and $b = GMat_0^2/(GMt_0^2 + 4\pi^2 a^3)$, from formula (12).

Equation (15) may be written as follows:

$$4\pi^2 b \cos^2\varphi\, c_x^3 - GMt_\varphi^2 c_x + GMt_\varphi^2 b = 0, \tag{16}$$

where the unknown quantity is c_x.

Taking the values of Kerrington [13] for the observed rotational periods of the sun-spots and zonal layers on the photosphere of the Sun, in days, and, at the corresponding latitudes, calculate the value of c_x by (16) and the difference $c-c_x$, in kilometres, where $c=ab/\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}$ is the radius-vector of the ideal ellipsoidal surface of the photosphere, we obtain the following results:

Table 2

φ	0°	10°	20°	30°	40°	50°	60°	70°	80°	85°	90°
t_φ	25.0	25.2	25.6	26.3	27.3	28.6	30.2	32.1	34.3	35	∞
$c-c_x$	0.0	0.2	0.6	1.13	1.78	1.4	1.32	0.76	0.32	0.18	0.0

As can be seen from Table 2, the figure of the photosphere of the Sun is outlined as a very slightly deformed ellipsoid with maximum difference of $c-c_x \simeq 1.78$ km at $\simeq \pm 40°$ from the ideal ellipsoidal surface.

As we see, the observed "equatorial acceleration" of the Sun, which is also a puzzle, comes as a direct consequence of the deformation of the equipotential surface of the photosphere, caused by the viscosity.

The results of the calculations of c_x by (16) and the difference $c-c_x$, in metres, for the Earth (earth-crust), with the accepted values of M, a, b and $t_\varphi = t_0 = 86164$ s, are given in Table 3.

Table 3

φ	0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
$t_\varphi = t_0$		t_0	t_0	t_0	t_0	t_0	t_0	t_0	t_0	t_0	t_0
$c-c_x$	0.0	3.0	13.5	27.4	28	29	28	20	12	1.7	0.0

Table 3 shows that the *common* figure of the Earth is outlined as a slightly deformed spheroid with maximum difference $c-c_x \simeq 29$ metres at $\simeq \pm 45°$ from the ellipsoidal surface. As is well known, the average deviation of the so-called normal spheroid of Clairaut from the surface of the two-axial ellipsoid of rotation, with the same semi-axes, is about 20 m [14].

All the above calculations are done in the system of units CGS.

The stratification (formation of bigger zonal layers) of the celestial body in the course of evolution, from the surface to the centre and from the equator to the pole is due, as we assume, to the viscosity and to the tendency of the particles of the body to rotate according to law (9). Such zonal layers are clearly seen in the atmosphere of Jupiter and Saturn, rotating with different angular velocities. In the Earth's upper atmosphere we also observe zonal layers and "jet streams", circulating from west to the east with a greater angular velocity.

Thanks to the great viscosity, the particles of the Earth's crust are rotating with equal (or almost equal) angular velocity. In the atmosphere of the Earth and that of the other planets, however, where the viscosity is much smaller, we have differential rotation. In the upper atmosphere of the

Earth, where the specific factors in the low atmosphere (relief, unequal heating of the Earth's surface on land and sea, at the equator and at the poles) do not play any role, we observe zonal winds of high velocity. It has been discovered by artificial satellites that the atmosphere at a height of 200–300 km rotates 1.3 times more rapidly than the Earth's crust [15]. A similar phenomenon is observed in the solar atmosphere [13]. We also know that the velocity of the uninterrupted general transport of the air and vapour masses from west to the east is higher than the velocity of the Earth's crust rotation. All these phenomena cannot be explained by the thermal factor only. Obviously, at the regular rotation of the zonal layers in the upper atmosphere of the planets and the stars, according to (9) and (15), the main role is played by gravitation, as at the orbital circulation of the planets, according to Kepler's laws.

Formula (15) can be written as follows:

$$V_{\varphi}^{*}=\sqrt{\frac{GM(c_x-b)}{c_x b}}, \tag{16}$$

where V_{φ}^{*} is the linear velocity of the zonal layers and zonal winds.

$$V_{\text{relative}}^{*}=V_{\varphi}^{*}-465\ \text{m/s}\times\cos\varphi.$$

6. Conclusion

The result (9), obtained by different methods of research, which very well describes the differential rotation of the young celestial bodies, can be interpreted as a common law operating under ideal conditions, in which gravitation only is playing the main role. The particular cases (12) and (14) of this law confirm its veracity and importance. Formula (12) is most simple and most exact, compared with the similar classical and contemporary results. The precision with which it gives the rotational period of the Earth can be explained by the great elasticity of the Earth, as a whole.

The result (15), where the influence of the viscosity is taken into account, could be successfully used, as we have shown, for determination of the common figure of the older celestial bodies, and for the explanation of their specific differential rotation.

We could give by means of (9) and (15) or (16) a qualitative explanation of the *general circulation* and the *dynamics* of the upper atmosphere of the Earth and other planets, assuming that the flattening $(a-b)/a$ of their compound envelopes increases with the height.

The experts in this subject could see, I believe, the significance of the results obtained in astrophysics and geophysics.

Acknowledgements. I am very grateful to Dr. M. Gogoshev and to Professor K. Serafimov for their kind support and to engineer Tihomir Dimitrov for valuable discussions in the course of this work.

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О дифференциальном вращении и фигуре небесных тел

Т. Р. Тилчев

(Резюме)

В этой статье дается теоретическое обоснование гипотезы автора о дифференциальном вращении небесных тел, опубликованной в журнале „Астрономише Нахрихтен“ профессором Никола Ст. Калициным [1]. Предложено оригинальное решение проблемы дифференциального вращения и фигуры небесных тел. Сформулированный общий закон действует при идеальных условиях. Тем не менее этот закон довольно хорошо описывает наблюдаемые явления в реальной природе. Первый частный случай этого закона сравнивается с подобными результатами Ньютона, Гюйгенса и Клеро. Второй частный случай является идентичным с третьим законам Кеплера для круговых орбит.